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Basis graphs of even Delta-matroids

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Abstract

A Δ-*matroid* is a collection $\mathcal{B}$ of subsets of a finite set I, called *bases*, not necessarily equicardinal, satisfying the symmetric exchange property: *For* $A, B \in \mathcal{B}$ *and* $i \in A \,\Delta\, B$, *there exists* $j \in B \,\Delta\, A$ *such that* $(A \,\Delta\, \{i, j\}) \in \mathcal{B}$. A Δ-matroid whose bases all have the same cardinality modulo 2 is called an *even* Δ-*matroid*. The *basis graph* $G = G(\mathcal{B})$ of an even Δ-matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by a single exchange (i.e., $|A \,\Delta\, B| = 2$). In this note, we present a characterization of basis graphs of even Δ-matroids, extending the description of basis graphs of ordinary matroids given by S. Maurer in 1973:

Theorem. *A graph* $G = (V, E)$ *is a basis graph of an even* Δ-*matroid if and only if it satisfies the following conditions*:

(a) *if* $x_1x_2x_3x_4$ *is a square and* $b \in V$, *then* $d(b, x_1) + d(b, x_3) = d(b, x_2) + d(b, x_4)$;
(b) *each* 2-*interval of* G *contains a square and is an induced subgraph of the* 4-*octahedron*;
(c) *the neighborhoods of vertices induce line graphs, or, equivalently, the neighborhoods of vertices do not contain induced* 5- *and* 6-*wheels.*

(A 2-*interval* is the subgraph induced by two vertices at distance 2 and all their common neighbors; a *square* is an induced 4-cycle of G.)

Keywords: Basis graph; Δ-Matroid; Graph distance

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1. Introduction

Matroids constitute an important unifying structure in combinatorics, algorithmics, and combinatorial optimization. According to one of the many equivalent definitions, a *matroid* on a finite set of *elements* I is a collection $\mathcal{B}$ of subsets of I, called *bases*, which satisfy the following *exchange property*:

(EP) *For all* $A, B \in \mathcal{B}$ *and* $i \in A \setminus B$ *there exists* $j \in B \setminus A$ *such that* $A \setminus \{i\} \cup \{j\} \in \mathcal{B}$.

We say that the base $A \setminus \{i\} \cup \{j\}$ is obtained from the base A by an *elementary exchange*. It is well known that all the bases of a matroid have the same cardinality, which is called its *rank*.

The *basis graph* $G = G(\mathcal{B})$ of a matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by a single exchange (i.e., $|A \,\Delta\, B| = 2$, where the *symmetric difference* of two sets A and B is written and defined by $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$). It has been shown by Cunningham (unpublished), Holzmann, Norton, and Tobey [18], and Maurer [19] that basis graphs faithfully represent their matroids, thus studying the basis graph amounts to studying the matroid itself (see [19] for some further references on basis graphs). Gelfand, Goresky, MacPherson, and Serganova [17] defined a basis matroid polyhedron as the convex hull of characteristic vectors of bases of a matroid. They showed that a convex hull of characteristic vectors of a collection $\mathcal{A}$ of equicardinal subsets of an n-element set is a basis matroid polytope if and only if its 1-skeleton is isomorphic to the basis graph of the family $\mathcal{A}$.

In [19], Maurer presented a characterization of graphs which are basis graphs of matroids and described basis graphs of several important classes of matroids, in particular, of binary matroids (see also [14] for a characterization of basis graphs of uniform matroids and [20] for investigation of properties of intervals of matroid basis graphs). From this characterization easily follows that all basis graphs are homotopically trivial, a property used several times in the theory of ordinary and oriented matroids; cf. [2,3,16]. From this result also follows that the 2-dimensional faces of a basis matroid polytope are equilateral triangles or squares; cf. [6].

There are several important and interesting generalizations of the concept of matroid. Delta-matroids is one of them, and have been introduced independently by Bouchet [9–11], Chandrasekaran and Kabadi [13], and Dress and Havel [15] in essentially equivalent ways. A Δ-*matroid* is a collection $\mathcal{B}$ of subsets of a finite set I, called *bases*, not necessarily equicardinal, satisfying the following *symmetric exchange property*:

(SEP) *For* $A, B \in \mathcal{B}$ *and* $i \in A \,\Delta\, B$, *there exists* $j \in B \,\Delta\, A$ *such that* $(A \,\Delta\, \{i, j\}) \in \mathcal{B}$.

It is immediately clear that the family of bases of a matroid is also a Δ-matroid. In fact, matroids are precisely the Δ-matroids for which all members of $\mathcal{B}$ have the same cardinality. A Δ-matroid whose bases all have the same cardinality modulo 2 is called an *even* Δ-*matroid*. If A, B are two bases of an even Δ-matroid $\mathcal{B}$ and $B = A \,\Delta\, \{i, j\}$ we say that B is obtained from A by an *elementary exchange*. Following the terminology for ordinary matroids, the *basis graph* $G = G(\mathcal{B})$ of an even Δ-matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by a single exchange, i.e., A and B are adjacent if and only if $|A \,\Delta\, B| = 2$. Some properties of these graphs have been used and investigated by Wenzel [22,23].

For a subset I' of I denote $\mathcal{B} \,\Delta\, I' := \{B \,\Delta\, I' \colon\ B \in \mathcal{B}\}$ and say that $\mathcal{B} \,\Delta\, I'$ is obtained by applying a *twisting* to $\mathcal{B}$. Then $\mathcal{B} \,\Delta\, I'$ is a Δ-matroid. It can be easily shown that the even Δ-matroids $\mathcal{B}$ and $\mathcal{B} \,\Delta\, I'$ have isomorphic basis graphs. Applying a twisting to a matroid we will always obtain an even Δ-matroid, but only in some cases this will be a matroid (for example, if $I' = I$, we will get the dual matroid).

It will be convenient to identify I with the set $[n] = \{1, 2, \ldots, n\}$. Let $I^* = \{1^*, 2^*, \ldots, n^*\}$. Bouchet defines a *symmetric matroid* as essentially a Δ-matroid with bases extended to n elements by adding to $B \in \mathcal{B}$ all starred elements from I^* which do not appear, unstarred, in B. Thus a symmetric matroid is a family $\mathcal{B}$ of subsets of cardinality n of the set $I \cup I^*$ such that in every subset of $\mathcal{B}$ each element of I appears either unstarred or starred and if

For $A, B \in \mathcal{B}$ and $i \in A \,\Delta\, B$, there exists $j \in B \,\Delta\, A$ such that $(A \,\Delta\, \{i, j, i^*, j^*\}) \in \mathcal{B}$.

Many important properties associated with matroids (greedy algorithm, polyhedral description) extend to Δ-matroids. Apart from well-known examples of matroids, several other discrete structures satisfy (SEP). For example, consider a skew-symmetric matrix $M = (m_{ij}\colon\, i, j \in I)$ (i.e., $M = (-M)^T$ and all diagonal entries of M are zero) and define $\mathcal{B}$ by letting $B \in \mathcal{B}$ if and only if the principal submatrix $(m_{ij}\colon\, i, j \in B)$ is non-singular. Then $\mathcal{B}$ is an even Δ-matroid [10]. Second, let $G = (V, E)$ be a graph, and a subset of vertices B of G belongs to $\mathcal{B}$ if and only if there is a matching of G covering precisely the elements of B. The resulting collection $\mathcal{B}$ is an even Δ-matroid [11,13]. Another nice instance of an even Δ-matroid is given in [4,12] and arises by considering a graph $G = (V, E)$ drawn on a compact surface S and its geometric dual $G^* = (F, E^*)$: $e^* \in E^*$ is the unique edge of G^* which cuts the edge $e \in E$. Let $\mathcal{B}$ consist of maximal by inclusion subsets B of $E \cup E^*$ (e and e^* cannot be simultaneously in B) such that the surface S is not disconnected by cutting it along the edges from B. Then $\mathcal{B}$ is an even Δ-matroid. Finally notice that Δ-matroids occupy an important place in the theory of Coxeter matroids (where one can find them under the name of "Lagrangian matroids") [5,7,8].

In this note, we characterize the basis graphs of even Δ-matroids, extending and refining Maurer's description of basis graphs of matroids. Our proof provides an alternative approach to Maurer's result: departing from a graph satisfying Maurer's conditions and performing a suitable twisting to the even Δ-matroid obtained in our proof, we get a matroid.

2. Terminology and main results

In this section we establish our notation and formulate the principal results. All graphs $G = (V, E)$ occurring here are finite, connected, and without loops or multiple edges. For brevity's sake, we use the notation $x \sim y$ if x, y are adjacent and $x \nsim y$ otherwise. If $x \sim y$, we denote the corresponding edge by xy. By a *square* $x_1x_2x_3x_4$ we will mean an induced 4-cycle with $x_1 \nsim x_3$ and $x_2 \nsim x_4$. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v of a graph $G = (V, E)$ is the length of a shortest path between u and v. The set of all vertices w on shortest (u, v)-paths is the *interval* $[u, v]$. For convenience, we will use the short-hand $(u, v) = [u, v] \setminus \{u, v\}$ to denote the "interior" of the interval between u and v. If $d(u, v) = 2$, we say that $[u, v]$ is a 2-*interval*. A subset of vertices S of G (or the subgraph $G(S)$ induced by S) is called *convex* if $[u, v] \subseteq S$ for any $u, v \in S$. The *convex hull* $\operatorname{conv}(A)$ of a set A is the smallest convex set containing A. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. It will be convenient to use the same notation and terminology for a subset of vertices and the subgraph induced by this subset; for example, $[u, v]$ will also denote the subgraph induced by the interval between u and v.

The *Cartesian product* $G = G_1 \,\square\, G_2$ of two graphs G_1 and G_2 has the pairs (x_1, x_2) as its vertices (with vertex x_i from G_i) and an edge between two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$

if and only if, for some $i \in \{1,2\}$, the vertices x_i and y_i are adjacent in G_i, and $x_j = y_j$ for the remaining $j \neq i$. The *sum* of two even Δ-matroids $\mathcal{B}_1$ and $\mathcal{B}_2$ defined on disjoint ground sets is the even Δ-matroid $\mathcal{B}_1 + \mathcal{B}_2 = \{B_1 \cup B_2\colon\ B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}$. In that case, we obtain $G(\mathcal{B}_1 + \mathcal{B}_2) = G(\mathcal{B}_1) \,\square\, G(\mathcal{B}_2)$.

Three important classes of graphs serve as host spaces for basis graphs in question. Denote by H_n the n-cube; notice that $d_{H_n}(A,B) = |A \,\Delta\, B|$ for any $A, B \in 2^{[n]}$. Let $[A,B]_{H_n}$ be the interval between A and B; clearly $[A,B]_{H_n} = \{C\colon\ A \cap B \subseteq C \subseteq A \cup B\}$. The *half-cube* $\frac{1}{2}H_n$ is the graph whose vertex set is the collection of all subsets of $[n]$ which have the same cardinality modulo 2 and two vertices A, B are adjacent in $\frac{1}{2}H_n$ iff $|A \,\Delta\, B| = 2$. Given two vertices A, B of $\frac{1}{2}H_n$, we denote by $[A,B]_{\frac{1}{2}H_n}$ all subsets of $[A,B]_{H_n}$ which are vertices of $\frac{1}{2}H_n$. The *Johnson graph* $J_{n,k}$ is the (isometric) subgraph of $\frac{1}{2}H_n$ induced by the family of all subsets of cardinality k. Axioms (EP) and (SEP) imply that the basis graphs of matroids of rank k are isometric subgraphs of $J_{n,k}$, while the basis graphs of even Δ-matroids are isometric subgraphs of $\frac{1}{2}H_n$. Recall also that the *m-octahedron* $K_{m\times 2}$ is the complete multipartite graph with $m \geqslant 2$ parts, each of size 2. Equivalently, $K_{m\times 2}$ is obtained from the complete graph K_{2m} by deleting a perfect matching. Notice that all 2-intervals in the half-cube $\frac{1}{2}H_n$ with $n \geqslant 4$ are 4-octahedra and all 2-intervals in the Johnson graph $J_{n,k}$ with $n \geqslant 4$ and $1 < k < n$ are 3-octahedra. Therefore 2-intervals $[A,B]$ in basis graphs of matroids and even Δ-matroids are connected induced subgraphs of 3-octahedra and 4-octahedra, respectively. Additionally, as we will show in Section 4, the exchange properties (EP) and (SEP) imply that the interior of $[A,B]$ is not a complete graph; for example, each 2-interval in the basis graph of a matroid is either an induced square, a pyramid with square base, or a 3-octahedron. We shall say that a graph G satisfies the *interval condition* (ICm) if

(ICm) *Every* 2-*interval* $[u,v]$ *contains a square* $K_{2\times 2}$ *and is an induced subgraph of the m-octahedron* $K_{m\times 2}$.

A *levelling* of a graph $G = (V,E)$ from a base-point b is a partition of V into the sets $N_i(b) = \{v \in V\colon\ d(b,v) = i\}$. We will denote by $N(b)$ the subgraph induced by $N_1(b)$. Define a partial order $\prec_b$ on V by letting $u \prec_b v$ iff $d(b,v) = d(b,u) + d(u,v)$. Maurer [19] established that a graph G is a matroid basis graph if and only if it satisfies (i) the interval condition (IC3), (ii) the neighborhood $N(b)$ of some vertex b is the line graph of a bipartite graph, and (iii) in any levelling of G each octahedral 2-interval lies in one of three positions: (1) all in one level; (2) in two levels, three adjacent vertices in each; or (3) in three levels, a square in between, one vertex in the highest, and one vertex in the lowest, any other 2-interval lies as an induced subgraph of a 3-octahedron positioned as above. We will use another (apparently simpler) *positioning condition*, which, as we will show in Section 4, it is satisfied by half-cubes and all their isometric subgraphs:

(PC) *For every vertex b and every square* $v_1v_2v_3v_4$ *of* G, *the following equality holds*:

$$d(b,v_1) + d(b,v_3) = d(b,v_2) + d(b,v_4).$$

We will call a square *upward* if it belongs to three consecutive levels, *horizontal* if it lies in one level, and *vertical* if two adjacent vertices are in one level and other two adjacent vertices are in the next level. An edge uv of G is *horizontal* if u and v lie in one level and *upward* otherwise. A k-*wheel* W_k is the graph consisting of a k-cycle plus an additional vertex adjacent to all vertices of the cycle. Finally, recall that the *line graph* $L(\Gamma)$ of a graph Γ has the edges of Γ as vertices and two vertices are adjacent in $L(\Gamma)$ if and only if the corresponding edges are incident in Γ.

We are ready to formulate the main results of this note.

Theorem 1. *For a graph $G = (V, E)$ the following conditions are equivalent*:

(i) *G is a basis graph of an even Δ-matroid*;
(ii) *G satisfies the interval condition* (IC4)*, the positioning condition* (PC)*, and no neighborhood $N(v)$ contains an induced* 5*-wheel W_5 or* 6*-wheel W_6*;
(iii) *G satisfies the interval condition* (IC4)*, the positioning condition* (PC)*, and the neighborhood of every vertex is a line graph*;
(iv) *G is a connected induced subgraph of a half-cube satisfying the interval condition* (IC4);
(v) *G is an isometric subgraph of a half-cube satisfying the interval condition* (IC4).

The proof of Theorem 1 uses several auxiliary results given in Section 3 and is presented in Section 4. The central issue of this proof is to establish that (iii) implies (iv). We briefly outline the idea of the encoding employed in this proof. Let $G = (V, E)$ be a graph satisfying (IC4) and (PC), such that for some vertex b, $N(b)$ is the line graph of a graph $\Gamma = (I, F)$. To establish that G is an induced subgraph of a half-cube, we define the following mapping $\varphi : V \to 2^I$. Set $\varphi(b) = \emptyset$. Each vertex $x \in N(b)$ encodes some edge ij of Γ; put $\varphi(x) = \{i, j\}$. For any other vertex v, let $\varphi(v) = \bigcup\{\varphi(x)\colon\ x \in [b, v] \cap N(b)\}$. We show that φ is injective and that all sets $\varphi(v)$ have even cardinality. Finally, we show that φ is an edge-preserving map from G to the half-cube $\frac{1}{2}H_n$, thus establishing (iii) $\Rightarrow$ (iv). From the proof of (iv) $\Rightarrow$ (v) $\Rightarrow$ (i), we conclude that the resulting collection $\mathcal{B}_\varphi := \{\varphi(v)\colon\ v \in V\}$ *is an even Δ-matroid whose basis graph is G*. The proof of the implication (ii) $\Rightarrow$ (iii) is based on the case analysis of forbidden subgraphs of line graphs and is postponed to the final Section 7.

We show that, if Γ is a bipartite graph with $I = A \cup B$, then $\mathcal{B}_\varphi\ \Delta\ B$ is a matroid of rank $|B|$, thus providing an alternative proof of Maurer's characterization. Our encoding scheme is different from that used by Maurer [19], except the encoding of the vertices of $N(b)$, where both schemes are essentially the same. Recall, he encodes the vertex b by B, a vertex $x \in N(b)$ representing the edge ij of Γ with $i \in B$ and $j \in A$ is labelled by the set $B \setminus \{i\} \cup \{j\}$. The encoding is inductively expanded to the whole graph using certain upward squares (among other things, in establishing that this labelling is well defined, it is necessary to show that it is independent of chosen squares); see [19] for all details. Notice also that for matroids, a result similar to the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 1 is given in Theorem 2.2 of [19].

In Section 5, we establish the following decomposition result of basis graphs:

Proposition 1. *Let G be a basis graph of an even Δ-matroid such that the neighborhood $N(b)$ of some vertex b is not connected. If $N(b)$ is the line graph of a graph $\Gamma = (I, F)$ and (I_1, F_1) is a connected component of Γ, then $\mathcal{B}_\varphi$ is the sum of the even Δ-matroids $\mathcal{B}_1 = \{B \in \mathcal{B}_\varphi\colon\ B \subseteq I_1\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B}_\varphi\colon\ B \subseteq I \setminus I_1\}$, and consequently $G = G(\mathcal{B}_1) \square G(\mathcal{B}_2)$.*

In view of this result, we call a basis graph *indecomposable* if the neighborhoods of all its vertices induce connected graphs. Define the *support* of an even Δ-matroid to be the elements of the ground set that actually appear in at least one base.

Proposition 2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two even Δ-matroids with supports I_1 and I_2, and isomorphic indecomposable basis graphs $G(\mathcal{B}_1) \simeq G(\mathcal{B}_2) \simeq G$. Then there exists a bijection $\alpha : I_1 \mapsto I_2$ and a set $S \subseteq I_2$ so that $\mathcal{B}_2 = \{\alpha(B) \,\Delta\, S\colon\ B \in \mathcal{B}_1\}$, unless G is an induced subgraph of the 4-octahedron.*

In fact, our proof yields something a little stronger. Let $B_i(v)$ denote the set in $\mathcal{B}_i$ represented by a vertex v of G, for $i = 1$ or 2. Then α and S have the property that $B_2(v) = \alpha(B_1(v)) \,\Delta\, S$ for every vertex v of G. In other words, unless the basis graphs are isomorphic to an induced subgraph of the 4-octahedron, every isomorphism between basis graphs (represented by the identifications of vertices of $G(\mathcal{B}_1)$ and $G(\mathcal{B}_2)$ with vertices of G) is induced by a bijection from one support to the other followed by a twisting.

In Section 6, we extend the characterization of Gelfand et al. [17] of basis matroid polyhedra to basis polyhedra of even Δ-matroids. Let $\mathcal{B} \subset 2^{[n]}$. For $B \in \mathcal{B}$, let $\sigma(B)$ be its incidence 0, 1-vector: $\sigma_i(B) = 1$ if $i \in B$ and $\sigma_i(B) = 0$ otherwise. Let $\Pi(\mathcal{B})$ be the convex hull (in the usual sense) of the points $\sigma(B)$, $B \in \mathcal{B}$. Denote by $\Pi_1(\mathcal{B})$ the 1-*skeleton* of $\Pi(\mathcal{B})$: it has the vertices of $\Pi(\mathcal{B})$ as the set of vertices and two vertices are adjacent in $\Pi_1(\mathcal{B})$ if and only if the (linear) segment connecting them is a 1-dimensional face of the polyhedron $\Pi(\mathcal{B})$. Notice that for any collection of even subsets $\mathcal{B}$, $\Pi_1(\mathcal{B})$ contains the subgraph $G(\mathcal{B})$ of $\frac{1}{2}H_n$ induced by its vertices. Indeed, it is easy to show explicitly that the set $\mathcal{E}$ of all even subsets of $\{1, 2, \ldots, n\}$ generates a polyhedron having the half-cube $\frac{1}{2}H_n$ as 1-skeleton. Then $G(\mathcal{B}) \subseteq \Pi_1(\mathcal{B})$ follows because any edge of $\Pi_1(\mathcal{E})$ with vertices in $\mathcal{B}$ must be an edge of $\Pi_1(\mathcal{B})$ also. As we will show below, the converse inclusion characterizes the even Δ-matroids:

Theorem 2. *If $\mathcal{B}$ is an even Δ-matroid, then the graph $\Pi_1(\mathcal{B})$ coincides with the basis graph $G(\mathcal{B})$. Conversely, if $\mathcal{B} \subset 2^{[n]}$ is a collection of sets of even cardinality and the graphs $\Pi_1(\mathcal{B})$ and $G(\mathcal{B})$ coincide, then $\mathcal{B}$ is an even Δ-matroid.*

3. Auxiliary results

In this section, we establish some results that will be used throughout the proof of Theorem 1. Unless stated otherwise, G is always a graph fulfilling condition (iii) of Theorem 1. For properties (3.1) and (3.2) in case of matroid basis graphs see also [19]. The proofs of (3.3) and (3.4) become much shorter if one replace (IC4) by (IC3).

(3.1) (Triangle condition) *If $u \sim v$ and $d(b, u) = d(b, v) = k$, then there exists a vertex $w \sim u, v$ at distance $k - 1$ to b (a common parent of u, v with respect to $\prec_b$).*

Proof. Proceed by induction on k. Pick $z \in [u, b]$ adjacent to u. If $z \sim v$, we are done. Otherwise take a square $zxvy$. Then $x \in N_k(b)$ and $y \in N_{k-1}(b)$ by (PC). If $x \neq u$, then u is adjacent to x and y by (IC4), therefore y is the desired parent. Finally, let $u = x$. By the induction hypothesis, there is a vertex $t \sim z, y$ at distance $k - 2$ to b. Consider an upward square containing v, t, and two other vertices in $N_{k-1}(b)$. By (PC), u must be adjacent to one of these vertices. □

A *propeller* is a graph obtained by gluing three 3-cycles along a common edge.

(3.2) (No propellers) *G does not contain induced propellers.*

Proof. Suppose G has a propeller with tips x_1, x_2, x_3, and the edge y_1y_2 whose ends are adjacent to all tips. Then $N(y_2)$ contains a $K_{1,3}$ subgraph induced by x_1, x_2, x_3, and y_1, which is forbidden in line graphs. □

(3.3) (Local convexity implies convexity) *A connected induced subgraph H of G is convex if and only if whenever $x, y \in H$ and $d(x, y) = 2$, then $[x, y] \subseteq H$.*

Proof. We will show that $[u, v] \subseteq H$ for $u, v \in H$ by induction on $k = d_H(u, v)$, the case $d_H(u, v) = 2$ being covered by the initial assumption. Suppose by way of contradiction that one can find $u, v \in H$ at distance $k \geqslant 3$ in H and a vertex $w \in [u, v] \setminus H$. Additionally assume that, if there are several such pairs, the selected one has least distance $d(u, v)$ in G. Pick a neighbor z of u on a shortest (u, v)-path P in H. Then $[v, z] \subseteq H$ by the induction hypothesis, in particular $d_H(v, z) = d(v, z)$. If $d(u, v) = d(z, v)$, by (3.1) u and z have a common neighbor in $[v, z]$, contrary to the choice of P. Thus $d(u, v) = d(z, v) + 1 = d_H(u, v)$. Let t be a neighbor of u in a shortest (u, v)-path passing via w. If $t \sim z$, by (3.1) they have a common neighbor $s \in [v, z] \subset H$. Since $u \nsim s$, we obtain $t \in [s, u] \subset H$. Then, by the choice of u, v, we have $[t, v] \subset H$, contrary to $w \in [t, v] \setminus H$. Thus $t \nsim z$. Consider a square ts_1zs_2. If u is a vertex of this cycle, say $u = s_2$, then $d(s_1, v) = k - 2$ by (PC), yielding $s_1 \in H$ because $s_1 \in [z, v]$. Since $t \in [u, s_1] \subset H$, we obtain the same contradiction as before. Hence $u \neq s_1, s_2$, whence $u \sim s_1, s_2$. Applying (PC) to v and the square s_1ts_2z we conclude that either $d(v, s_1) = d(v, s_2) = k - 1$ or $d(v, s_1) = k - 2$ and $d(v, s_2) = k$. In the second case, we could replace s_2 by u and have the case where u belongs to the square ts_1zs_2. Thus $d(v, s_1) = d(v, s_2) = k - 1$. If $s_1, s_2 \in H$, then $t \in [s_1, s_2] \subset H$. By the choice of u, v, each of the intervals $[s_1, v], [s_2, v]$, and $[t, v]$ belong to H, a contradiction. Thus, let $s_1 \notin H$. By (3.1), there exists a vertex $p \sim s_1, z$ at distance $k - 2$ to v. Since $p \in [z, v] \subset H$ and $s_1 \in [u, p]$, we get a contradiction with local convexity of H. □

(3.4) *The intervals of G are convex.*

Proof. By (3.3) it suffices to establish the local convexity of intervals. Suppose by way of contradiction that there exist $u, v \in V$, $x, y \in [u, v]$ with $d(x, y) = 2$, and a vertex $z \in (x, y) \setminus [u, v]$. By (PC) all 2-intervals are convex, thus $d(u, v) \geqslant 3$. Further, we may assume that $[u, x] \cap [u, y] = \{u\}$, otherwise u can be replaced by a closest to x and y vertex in this intersection. Analogously, $[v, x] \cap [v, y] = \{v\}$. We distinguish two cases:

Case 1. $d(u, x) < d(u, y)$, $d(v, x) > d(v, y)$.

Then $d(u, z) = d(u, x) + 1$, $d(v, z) = d(v, y) + 1$, and $d(u, y) = d(u, x) + 1$, $d(v, x) = d(v, y) + 1$, otherwise $z \in [u, v]$. Pick a square $xx'yy'$. By (PC), one can assume that $d(u, x') = d(u, x)$, $d(v, y') = d(v, y)$. Hence $x', y' \in [u, v]$ and $z \sim x', y'$. The choice of u, v and (3.1) yield that $u \sim x, x'$ and $v \sim y, y'$, hence $d(u, v) = 3$.

First suppose that there exists a vertex $a \in [u, y]$, $a \nsim x'$, $a \neq x'$. Then $a \sim y'$ and $a \nsim z$ by (PC). Since $d(u, v) = 3$, a is not adjacent to v, hence y, y', a, z, v induce a propeller. Thus there exist distinct vertices $a', a'' \in [u, y]$ and $b', b'' \in [x, v]$, such that $a' \nsim a''$, $b' \nsim b''$, $x' \sim a', a''$, and $y \sim b', b''$. By (PC), y' is adjacent to one of the vertices a', a'', say $y' \sim a'$. To avoid propellers, we must have $a' \sim z$, further $a' \sim x$ by (IC4). Similarly, one of the vertices b', b'', say b', is adjacent to x', z, y. If $a' \sim b'$, then (x, y) contains a K_4 induced by x', a', b', z. Thus $a' \nsim b'$, whence $a' \sim b''$ and $b' \sim a''$ by (PC). Since $z \nsim u$, $b'' \nsim u$, and $b'' \nsim z$ by (PC), the vertices a', x, u, z, b'' induce a propeller.

Case 2. $d(u, x) = d(u, y) =: k$, $d(v, x) = d(v, y) =: m$.

Pick a square xz_1yz_2. By (PC), either $d(u,z_1)=k-1$, $d(u,z_2)=k+1$ and $d(v,z_2)=m-1$, $d(v,z_1)=m+1$, or $d(u,z_1)=d(u,z_2)=k$ and $d(v,z_1)=d(v,z_2)=m$. In both cases we obtain $z_1,z_2\in[u,v]$, thus $z\sim z_1,z_2$. In the first case we have $d(u,z)=k$, $d(v,z)=m$, implying that $z\in[u,v]$. Thus only the second possibility may occur. By (3.1) one can find vertices $u',v'\sim x,z_1$ with $d(u,u')=k-1$ and $d(v,v')=m-1$. To avoid a propeller induced by x,z,z_1,u',v', the vertex z must be adjacent to one of u',v', say $z\sim u'$. Then $z\nsim v'$. If $u'\nsim y$, then $z\in[u',y]$, and we are in conditions of Case 1. Thus $u'\sim y$, hence $u=u'$ by the choice of u and v. Consequently, $u\sim z_2$. Applying (PC) to v' and the square xz_1yz_2 we conclude that either $v'\sim y,z_2$ or $v'\nsim y,z_2$. If v' is adjacent to y and z_2, then $[x,y]$ violates (IC4) since $v',u,z\in[x,y]$ however $v'\nsim u,z$. Thus $v'\nsim y,z_2$. Consider a square $ysv'v''$. Then $d(v'',v)=m-1$, $d(s,v)=m$ by (PC). Hence $u\nsim v''$, yielding $u\sim s$. By (PC) each of the vertices x,z is adjacent to exactly one of the vertices s,v''. If $z\sim v''$, then $z\in[u,v]$ since $z\sim u,v''$ and $d(u,v)=d(v'',v)+2$. Thus $z\sim s$ and $z\nsim v''$. On the other hand, if $x\sim v''$, then $u,z,v''\in[x,y]$ however $v''\nsim z,u$, contrary to (IC4). Thus $x\sim s$ and $x\nsim v''$. If $s\neq z_1$, then $z_1\sim s$ and (x,y) will contain a 4-clique induced by u,z,z_1,s. So $z_1=s$. Therefore $z_2\sim v''$ by (PC). From (3.1) and the choice of u,v one conclude that $v\sim v',v''$, whence $d(u,v)=3$. Applying (IC4) for $[z,v']$ and (PC), we conclude that there exists a new vertex $t\in[z,v']$ adjacent to v''. Since t is adjacent to at least one of the vertices x,z_1, from Case 1 we infer that $t\in[u,v]$. Thus t is adjacent to u or v. If $t\sim v$, then either $t\sim x,y$ and the 2-interval $[u,t]$ is not convex, or $z\in[t,x]\cup[t,y]$ and we are in conditions of Case 1. Thus $t\sim u$. Then $z_2\sim t$ and $t\sim x$, because the 2-intervals $[u,v'']$ and $[z_2,v']$ are convex. If $t\nsim z_1$, then $z,u,v'\in[t,z_1]$ and $v'\nsim u,z$, contrary to (IC4). Otherwise, if $t\sim z_1$, then $t\sim y$ because $[x,y]$ is convex. But in this case (x,y) contains a K_4 induced by u,z,z_1,t. This concludes the analysis of Case 2, thus establishing the convexity of $[u,v]$. □

4. Proof of Theorem 1

We commence by the proof of Theorem 1.

(i) ⇒ (ii) and (iii): Let $\mathcal{B}\subset 2^{[n]}$ be an even Δ-matroid. To show that $G(\mathcal{B})$ satisfies (IC4), pick two bases A,B at distance 2. Applying a twisting, one can suppose that $A=\emptyset$, $B=\{1,2,3,4\}$. Since $[A,B]_{\frac{1}{2}H_n}$ is a 4-octahedron and $[A,B]\subseteq[A,B]_{\frac{1}{2}H_n}$, the second half of (IC4) follows. It remains to show that (A,B) contains two non-adjacent bases. By (SEP) one can find a base $C\in(A,B)$, say $C=\{1,2\}$. Applying (SEP) to the bases A,B and the elements $1,2$, we will get either two complementary bases C',C'' of cardinality 2 each (i.e., $C'\cap C''=\emptyset$, $C'\cup C''=B$) or the bases $\{1,i\},\{2,i\}$ for some $i\in\{3,4\}$, say $i=3$. Now, applying (SEP) to A and the element 4, one conclude that $\{j,4\}\in\mathcal{B}$ for some $j\in\{1,2,3\}$. Obviously, this base and one of previously defined bases are complementary, establishing the first part of (IC4).

Using the induction and axiom (SEP), one can easily show that $G(\mathcal{B})$ is an isometric subgraph of the half-cube $\frac{1}{2}H_n$. To establish (PC), it suffices to verify this condition for some base-point B and the square $A_1A_2A_3A_4$, where $A_1=\emptyset$, $A_2=\{1,2\}$, $A_3=\{1,2,3,4\}$, and $A_4=\{3,4\}$ (again twisting does the job). Obviously $|B|+|B\,\Delta\,\{1,2,3,4\}|=|B\,\Delta\,\{1,2\}|+|B\,\Delta\,\{3,4\}|$. Since the distances $d(B,A_i)$ in $G(\mathcal{B})$ and $\frac{1}{2}H_n$ are equal, we have $d(B,A_1)+d(B,A_3)=d(B,A_2)+d(B,A_4)$. This shows that $G(\mathcal{B})$ satisfies the positioning condition (PC).

The neighborhood of every vertex of the half-cube $\frac{1}{2}H_n$ is the line graph of the complete graph K_n. Therefore the neighborhood of every vertex A of an arbitrary induced subgraph G of $\frac{1}{2}H_n$ is the line graph of some subgraph Γ_A of K_n: if say $A=\emptyset$, then Γ_A consists of all pairs

ij such that $\{i, j\}$ is a vertex of G. So we have established (iii), and (ii) follows from the simple observation that neither W_5 nor W_6 can occur as an induced subgraph of a line graph.

(iii) $\Rightarrow$ (iv): Assume that a graph G obeys the conditions (IC4) and (PC). Additionally, suppose that there is a vertex b whose neighborhood $N(b)$ is the line graph of a graph $\Gamma = (I, F)$ with $I = \{1, 2, \ldots, n\}$. Let $I^* = \{1^*, 2^*, \ldots, n^*\}$. Define the following mapping $\varphi: V \to 2^I$. Take b as a base-point and set $\varphi(b) = \emptyset$. Each vertex $x \in N(b)$ encodes some edge ij of Γ; put $\varphi(x) = \{i, j\}$. For a vertex $v \nsim b$, set $C_v := N(b) \cap [b, v]$ and $\varphi(v) = \bigcup\{\varphi(x)\colon x \in C_v\}$. Clearly φ is isotone with respect to the base-point order $\prec_b$: $u \prec_b v$ implies $\varphi(u) \subseteq \varphi(v)$. For a vertex $v \in V$, let $\varphi^*(v) = \{i^*\colon i \notin \varphi(v)\}$. Given an index $a \in I \cup I^*$, define

$$W_a = \bigl\{v \in V\colon a \in \varphi(v) \cup \varphi^*(v)\bigr\}.$$

To show that φ embeds the graph $G = (V, E)$ as an induced subgraph of the half-cube $\frac{1}{2}H_n$, it suffices to establish that φ is (1) injective (i.e., $\varphi(u) = \varphi(v)$ if and only if $u = v$), and (2) edge-preserving (i.e., $|\varphi(u) \,\Delta\, \varphi(v)| = 2$ if and only if $u \sim v$ in G). Since $\varphi(b) = \emptyset$ and G is connected, from (2) would follow that all sets $\varphi(v)$, $v \in V$, have even cardinality. We start with some properties of the map φ and sets W_a, $a \in I \cup I^*$.

(4.1) *If $x, y \in C_v$ and $\varphi(z) \subseteq \varphi(x) \cup \varphi(y)$ for $z \in N(b)$, then $z \in C_v$.*

Proof. Clearly $z \sim x, y$. Suppose $z \notin [b, v]$. Then $x \sim y$, because $[b, v]$ is convex. One can assume that $\varphi(x) = \{1, 2\}$, $\varphi(y) = \{1, 3\}$, $\varphi(z) = \{2, 3\}$. By (3.1), there exists a common neighbor v_0 of x and y one step closer to v. Then $z \nsim v_0$, otherwise $z \in [v_0, b] \subseteq [v, b]$. Consider a square v_0sbt. If it contains x or y, say $t = x$, then $y \sim s$ and $z \nsim s$ by (PC). Since $s \sim y$, the label of s must contain 1 or 3, which is impossible because s is not adjacent to x and z. So $s, t \neq x, y$, therefore $x, y \sim s, t$ by (IC4). By (PC) one can assume that $z \sim t$, $z \nsim s$. But then the vertex t cannot be labelled: on the one hand, $\varphi(t) \subset \{1, 2, 3\}$, however all three 2-subsets of this set have been used already to label the vertices x, y, z. □

(4.2) *The map $\varphi: V \to 2^I$ is injective.*

Proof. The proof is a consequence of the following properties:

(a) for every vertex $v \nsim b$ we have $[v, b] = \operatorname{conv}(C_v)$;
(b) if $\varphi(u) \subseteq \varphi(v)$, then $u \prec_b v$.

To establish (a), proceed by induction on $d(v, b)$. Let $wtvs$ be an upward square with $w \prec_b v$. If $d(v, b) = 2$, then $w = b$ and $s, t \in C_v$. Since $b, v \in [s, t]$, the result follows. If $d(v, b) > 2$, by the induction assumption we have $[s, b] = \operatorname{conv}(C_s)$ and $[t, b] = \operatorname{conv}(C_t)$. Since $C_s \cup C_t \subseteq C_v$ and $v \in [s, t]$, necessarily $v \in \operatorname{conv}(C_v)$, whence $[v, b] \subseteq \operatorname{conv}(C_v)$. On the other hand, since $[v, b]$ is convex and $C_v \subset [v, b]$, we obtain the required equality. To prove (b), suppose that $\varphi(u) \subseteq \varphi(v)$. We assert that $C_u \subseteq C_v$, with equality if $\varphi(u) = \varphi(v)$. Then (a) would imply that $u \prec_b v$ (and $u = v$ if $\varphi(u) = \varphi(u)$). So, let $z \in C_u \setminus C_v$. Since $\varphi(u) \subseteq \varphi(v)$, there exist two vertices $x, y \in C_v$ such that $\varphi(z) \subseteq \varphi(x) \cup \varphi(y)$. By (4.1) we have $z \in C_v$, contrary to the choice of z. □

(4.3) *For $a \in I \cup I^*$, an upward square $S = v_1v_2v_3v_4$ belongs to W_a whenever two opposite vertices of S belong to W_a. In particular, $\varphi(v_1) \cup \varphi(v_3) = \varphi(v_2) \cup \varphi(v_4)$.*

Proof. Suppose a is 1 or 1^*. Let $v_1, v_3 \in N_k(b)$, $v_4 \in N_{k-1}(b)$, and $v_2 \in N_{k+1}(b)$. If $v_2, v_4 \in W_1$, then $1 \in \varphi(v_4) \subseteq \varphi(v_1) \cap \varphi(v_3)$, whence $v_1, v_3 \in W_1$. Similarly, if $v_2, v_4 \in W_{1^*}$, then $v_1, v_3 \in W_{1^*}$, otherwise $1 \in \varphi(v_1) \cup \varphi(v_3) \subseteq \varphi(v_2)$. It remains to consider the cases $v_1, v_3 \in W_1$ and $v_1, v_3 \in W_{1^*}$.

Case 1. $v_1, v_3 \in W_1$.

Then $1 \in \varphi(v_2)$. To show that $1 \in \varphi(v_4)$, pick $x_1 \in C_{v_1}$ and $x_3 \in C_{v_3}$ with $1 \in \varphi(x_1) \cap \varphi(x_3)$. If $x_1 = x_3$, then $v_1, v_3 \in [v_2, b]$, $v_4 \in [v_1, v_3]$, hence $d(x_1, v_4) = k - 2$ by convexity of $[v_2, b]$. Thus $x_1 \in [v_4, b]$, yielding that $1 \in \varphi(v_4)$. Now, let $x_1 \neq x_3$ and $d(x_1, v_3) = d(x_3, v_1) = k$. Applying (PC) to the square S and each of the vertices x_1, x_3, we deduce that $d(x_1, v_4) = d(x_3, v_4) = d(b, v_4) = k - 1$. By (3.1) there exists a vertex $x_4 \sim x_1, b$ at distance $k - 2$ to v_4. Since $[v_3, b]$ is convex and $x_1 \notin [v_3, b]$, necessarily $x_4 \sim x_3$. Suppose $1 \notin \varphi(x_4)$. One can assume without loss of generality that $\varphi(x_1) = \{1, 2\}$, $\varphi(x_3) = \{1, 3\}$, $\varphi(x_4) = \{2, 3\}$. Since $x_1, x_4 \in [b, v_1]$ and $\varphi(x_3) \subset \varphi(x_1) \cup \varphi(x_4)$, from (4.1) we have $x_3 \in [b, v_1]$, which is impossible because $d(x_3, v_1) = k = d(b, v_1)$.

Case 2. $v_1, v_3 \in W_{1^*}$.

Then $v_4 \in W_{1^*}$ by definition of φ. Suppose by way of contradiction that $1 \in \varphi(v_2)$ and pick a vertex $x_2 \in C_{v_2}$ with $1 \in \varphi(x_2)$. If $d(x_2, v_4) = k - 1$, then $\min\{d(x_2, v_1), d(x_2, v_3)\} = k - 1$ by (PC), therefore $x_2 \in [b, v_1] \cup [b, v_3]$. Consequently $1 \in \varphi(v_1) \cup \varphi(v_3)$, a contradiction. So $d(x_2, v_4) = k$, and from (PC) we infer that $d(x_2, v_1) = d(x_2, v_3) = k$. By (3.1) there exist vertices $x_1 \sim x_2, b$ and $x_3 \sim x_2, b$ at distance $k - 1$ from v_1 and v_3, respectively. If $d(x_1, v_3) = k - 1$, then $d(x_1, v_4) = k - 2$ by (PC) applied to x_1 and $v_1v_2v_3v_4$. This implies $d(x_2, v_4) = k - 1$, a contradiction. Thus $x_1 \neq x_3$, moreover $x_1 \notin [b, v_3]$, $x_3 \notin [b, v_1]$. Let $\varphi(x_2) = \{1, 2\}$. Since $1 \notin \varphi(x_1) \cup \varphi(x_3)$ and $x_2 \sim x_1, x_3$, we conclude that $2 \in \varphi(x_1) \cap \varphi(x_3)$, hence $x_1 \sim x_3$. Let $\varphi(x_1) = \{2, 3\}$, $\varphi(x_3) = \{2, 4\}$. By Case 1 there is a vertex $x_4 \in C_{v_4}$ with $2 \in \varphi(x_4)$. Since $d(v_4, x_4) = k - 2$, x_4 is distinct from x_1, x_2, x_3. On the other hand, $x_4 \sim x_1, x_2, x_3$ because 2 belongs to the labels of all four vertices. Then $d(x_2, v_4) = k - 1$, which is impossible. □

To prove assertions (4.4)–(4.6), we proceed by induction on $k = d(b, v)$.

(4.4) *If uv is an upward edge of G and $u \prec_b v$, then $|\varphi(v) \setminus \varphi(u)| = 2$, and vice versa, if $\varphi(u) \subset \varphi(v)$ and $|\varphi(v) \setminus \varphi(u)| = 2$, then uv is an upward edge of G.*

Proof. The result is trivial if $u = b$. So, let $u \neq b$. Pick a neighbor w of u in $[u, b]$ and an upward square $wxvy$. By the induction assumption and (4.3) we may assume that $\varphi(x) = \varphi(w) \cup \{1, 2\}$, $\varphi(y) = \varphi(w) \cup \{3, 4\}$, $\varphi(v) = \varphi(w) \cup \{1, 2, 3, 4\}$. If u coincides with x or y, the result is immediate. Now, let $u \neq x, y$, thus $u \sim x, y$. Since $\varphi(w) \subset \varphi(u) \subset \varphi(v)$ and $|\varphi(u) \setminus \varphi(w)| = 2$, we conclude that $\varphi(u) = \varphi(w) \cup \{i_1, i_2\}$ for some $i_1 \in \{1, 2\}$ and $i_2 \in \{3, 4\}$. Conversely, since the map φ is isotone, one can easily deduce that $|\varphi(v)| = 2d(b, v)$ for every $v \in V$. Now, if $\varphi(u) \subset \varphi(v)$, then $u \prec_b v$ by assertion (b) in (4.2). In particular, if $|\varphi(v) \setminus \varphi(u)| = 2$, then v and u lie in consecutive levels, thus $u \sim v$. □

From (4.4) we conclude that φ maps the vertices of G to subsets of even cardinality, moreover the images of all vertices in the kth level have cardinality $2k$. One can further observe that $|\varphi(v) \setminus \varphi(u)| = 2d(v, u)$ whenever $u \prec_b v$.

(4.5) *If $i \in \varphi(v)$, then there exists a neighbor $x \in [v, b]$ of v such that $i \notin \varphi(x)$.*

Proof. Pick a neighbor $t \in [v, b]$ of v. Suppose $i \in \varphi(t)$, otherwise is nothing to show. By the induction hypothesis there is a vertex $w \in [t, b]$, $w \sim t$, such that $i \notin \varphi(w)$. Take an upward square $vxwy$ containing v, w. Since $i \notin \varphi(w)$, by (4.3) the label of one of the vertices x, y does not contain the index i. □

(4.6) *If $u, v \in N_k(b)$, then uv is a (horizontal) edge of G if and only if $|\varphi(u) \,\Delta\, \varphi(v)| = 2$.*

Proof. We may assume that $k \geqslant 2$, since the case $k = 1$ follows from the assumption that $N(b)$ is a line graph. Take $u, v \in N_k(b)$ fulfilling either one of two reciprocal conditions. If $u \sim v$, by (3.1) they have a common parent t. Otherwise, if $\varphi(v) = \varphi(u) \setminus \{1\} \cup \{2\}$, let $t \in N_{k-1}(b)$ be a neighbor of u such that $1 \notin \varphi(t)$ (its existence follows from (4.5)). Then $\varphi(t) \subset \varphi(v)$, thus by (4.4) t is adjacent also to v. In both cases, let w denote a neighbor of t in $N_{k-2}(b)$. Consider two upward squares $S' = wx'uy'$ and $S'' = wx''vy''$. From (4.3) we know that $\varphi(u) = \varphi(x') \cup \varphi(y')$ and $\varphi(v) = \varphi(x'') \cup \varphi(y'')$.

First, assume that $u \sim v$. If t belongs to both S' and S'', say $y' = t = y''$, then (PC) implies that $u \nsim x''$, $v \nsim x'$ and $x' \sim x''$. Therefore $\varphi(u) \,\Delta\, \varphi(v) = \varphi(x') \,\Delta\, \varphi(x'')$, and we are done because $|\varphi(x') \,\Delta\, \varphi(x'')| = 2$ by the induction hypothesis. Otherwise, if $t \neq x', y'$, then $t \sim x', y'$. By (PC), v is adjacent to x' or y', say $v \sim x'$. Applying the induction hypothesis, without loss of generality suppose $\varphi(x') = \varphi(w) \cup \{1, 2\}$, $\varphi(y') = \varphi(w) \cup \{3, 4\}$, and $\varphi(t) = \varphi(w) \cup \{1, 3\}$. By (4.3), $\varphi(u) = \varphi(w) \cup \{1, 2, 3, 4\}$, while the fact that φ is isotone and $|\varphi(v)| = |\varphi(u)|$ implies that $\varphi(v) = \varphi(w) \cup \{1, 2, 3, i\}$, for $i \notin \varphi(w) \cup \{4\}$. Hence $|\varphi(u) \,\Delta\, \varphi(v)| = 2$.

As to the converse, assume without loss of generality that $\varphi(u) = \varphi(w) \cup \{1, 3, 4, 5\}$ and $\varphi(v) = \varphi(w) \cup \{2, 3, 4, 5\}$, however $u \nsim v$. Since $1, 2 \notin \varphi(t)$, one can suppose that $\varphi(t) = \varphi(w) \cup \{3, 4\}$. If $y' = t = y''$, by (4.3) we deduce that $\varphi(x') \cap \varphi(x'') = \varphi(w) \cup \{5\}$. Hence $x' \sim x''$, but then the square S' and the vertex v violate the positioning condition. Otherwise, if, say, $t \notin S'$, then $t \sim x', y'$. If $1 \notin \varphi(x')$ say, then $\varphi(x') \subset \varphi(v)$, therefore $v \sim x'$ by (4.4). But then the vertices u, v, w, t, x' induce a propeller. □

Summarizing, from (4.2), (4.3), and (4.6) we deduce that $\varphi(G)$ is an induced subgraph of the half-cube $\frac{1}{2}H_n$, concluding the proof of (iii) ⇒ (iv).

(iv) ⇒ (v): We proceed by induction on the distance $k = d(A, B)$ between A, B in G. If $k = 1$ or 2 we are done, because G is an induced subgraph of $\frac{1}{2}H_n$ and $d_{\frac{1}{2}H_n}(A, B) \leqslant d(A, B)$. If $k \geqslant 3$, take a shortest (A, B)-path in G, $(A, C_1, \ldots, C_{k-2}, C_{k-1}, B)$. By (IC4) for G there exists a square $BC'C_{k-2}C''$ in $[B, C_{k-2}]$. Then $d(A, C') = d(A, C'') = k - 1$ and by induction hypothesis these distances are correct in $\frac{1}{2}H_n$, as is $d(A, C_{k-2}) = k - 2$. Then by (PC) for $\frac{1}{2}H_n$ applied to A and the square we get $d_{\frac{1}{2}H_n}(A, B) = k$, as required.

(v) ⇒ (i): It suffices to verify (SEP) for sets of the form $B = \emptyset$ and $A = \{1, 2, \ldots, r\}$. Further, (SEP) will follow if one can show that for every $i \in A$ there exists $j \in A$, $j \neq i$, such that $A \setminus \{i, j\}$ is a vertex of G. We proceed by induction on $d(A, B)$. Since G is an isometric subgraph of $\frac{1}{2}H_n$, there exists $k, l \in A$ such that $A' = A \setminus \{k, l\}$ is a vertex of G. If $i \in \{k, l\}$, then we are done. Otherwise, since $d(A', B) < d(A, B)$ and $i \in A'$, by the induction hypothesis there exists $m \in A'$ such that $A'' := A' \setminus \{i, m\}$ is a vertex of G. By (IC4) for G there exists a square $A''C'AC''$. Since the set $A'' = A \setminus \{i, k, l, m\}$ is contained in both C', C'' and $|C' \,\Delta\, C''| = 4$, we conclude that $C' \,\Delta\, C'' = \{i, k, l, m\}$. Thus i belongs to exactly one of the sets C', C'', say $i \in C'$. Then C'' is the required vertex of G. Hence the collection of sets defined by the vertices of G is an even Δ-matroid $\mathcal{B}$. Obviously, the basis graph of $\mathcal{B}$ is G. This concludes the proof of Theorem 1.

From the proof of Theorem 1 we infer that the encoding φ of $G=(V,E)$ defines an even Δ-matroid:

Corollary 1. *For a graph $G=(V,E)$ satisfying condition* (iii) *of Theorem* 1*, the collection $\mathcal{B}_\varphi=\{\varphi(v)\colon v\in V\}$ is an even Δ-matroid for any base-point $b\in V$ and $G=G(\mathcal{B}_\varphi)$.*

Using Theorem 1 and its proof, we obtain the following characterizations of basis graphs of ordinary matroids:

Corollary 2. *For a graph $G=(V,E)$ the following conditions are equivalent*:

(i) *G is a basis graph of a matroid*;
(ii) [19] *G satisfies the interval condition* (IC3)*, the positioning condition* (PC)*, and the neighborhood of every vertex is a line graph of a bipartite graph*;
(iii) *G is a basis graph of an even Δ-matroid and the neighborhood of every vertex is a line graph of a bipartite graph.*

Proof. The proof of (i) $\Rightarrow$ (ii) is analogous to the proof of (i) $\Rightarrow$ (ii) and (iii) of Theorem 1. To establish that (ii) $\Rightarrow$ (iii), notice that a graph G satisfying condition (ii) of Corollary 2 also satisfies condition (iii) of Theorem 1, thus G is a basis graph of an even Δ-matroid. Finally, assume that G satisfies (IC4), (PC), and the neighborhood $N(b)$ of the base-point b is the line graph of a bipartite graph $\Gamma=(A\cup B;F)$. We assert that $\mathcal{B}_\varphi\ \Delta\ B$ is a matroid. Evidently, it suffices to show that all the sets $\varphi'(v):=\varphi(v)\ \Delta\ B$ have size $k=|B|$. We proceed by induction on $d(b,v)$. The assertion is obviously true for b and the vertices in $N(b)$. Otherwise, if $v\notin N(b)$, pick an upward square $uxvy$. By induction hypothesis, the sets $\varphi'(u),\varphi'(x)$, and $\varphi'(y)$ have cardinality k. Then by (PC) for $\frac{1}{2}H_n$ applied to $\emptyset$ and the square $\varphi'(u)\varphi'(x)\varphi'(v)\varphi'(y)$ we get $|\varphi'(v)|=k$, as required. $\square$

Maurer [19] conjectured that the condition on neighborhoods of vertices in his characterization is redundant. In support of this conjecture, he established [19, Corollary 3.3] that (IC3) and (PC) imply that every neighborhood $N(b)$ is a line graph. In conjunction with our Theorem 1, this leads to the following result:

Corollary 3. *Any graph G satisfying the interval condition* (IC3) *and the positioning condition* (PC) *is the basis graph of an even Δ-matroid.*

Notice that the analogy of Maurer's conjecture for even Δ-matroids is false: in Section 7, we present two graphs satisfying (IC4) and (PC) in which the neighborhoods of several vertices are not line graphs.

5. Proof of Propositions 1 and 2

Proof of Proposition 1. Let $N(b)$ be the line graph of a graph $\Gamma=(I,F)$ having at least two connected components and let $\Gamma_1=(I_1,F_1)$ be one such component. Pick b as a base-point and consider the even Δ-matroid $\mathcal{B}_\varphi$. Set $\mathcal{B}_1=\{B\in\mathcal{B}_\varphi\colon B\subseteq I_1\}$ and $\mathcal{B}_2=\{B\in\mathcal{B}_\varphi\colon B\subseteq I\setminus I_1\}$. The graphs $G_1:=G(\mathcal{B}_1)$ and $G_2:=G(\mathcal{B}_2)$ induce convex subgraphs of the graph $G=G(\mathcal{B}_\varphi)$: given two vertices A,B of G_i, a vertex C of G belongs to $[A,B]$ if and only if $A\cap B\subseteq C\subseteq$

$A \cup B$, hence C belongs to G_i. Therefore both G_1 and G_2 satisfy condition (IC4), whence from Theorem 1 we infer that $\mathcal{B}_1, \mathcal{B}_2$ are even Δ-matroids and G_1, G_2 are their basis graphs.

To prove that $G = G_1 \square G_2$ it suffices to show that for any two bases $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, the set $B_1 \cup B_2$ is a base of $\mathcal{B}_\varphi$. We proceed by induction on $|B_1| + |B_2|$. By (SEP) applied to $B_1, \emptyset$ and $B_2, \emptyset$ we conclude that there exist $i, j \in B_1$ and $k, l \in B_2$ such that $B_1' := B_1 \setminus \{i, j\} \in \mathcal{B}_1$ and $B_2' := B_2 \setminus \{k, l\} \in \mathcal{B}_2$. By virtue of induction hypothesis, the sets $B_0 := B_1' \cup B_2' = (B_1 \cup B_2) \setminus \{i, j, k, l\}$, $B' := B_1 \cup B_2' = (B_1 \cup B_2) \setminus \{k, l\}$, and $B'' := B_1' \cup B_2 = (B_1 \cup B_2) \setminus \{i, j\}$ are bases of $\mathcal{B}_\varphi$. By (IC4) applied to $\mathcal{B}_\varphi$, there exist two bases A, B which together with B' and B'' form a square of $G(\mathcal{B}_\varphi)$. Then $A \cup B = B' \cup B'' = B_1 \cup B_2$. If $A = B_0$, then $B = B' \cup B''$, and we are done. So, let $A = (B' \cup B'') \setminus \{i, k\} = (B_1 \cup B_2) \setminus \{i, k\}$ and $B = (B' \cup B'') \setminus \{j, l\} = (B_1 \cup B_2) \setminus \{j, l\}$. Since $B \,\Delta\, B_1 = \{j\} \cup (B_2 \setminus \{l\})$ and $B \,\Delta\, B_1' = \{i\} \cup (B_2 \setminus \{l\})$, the base B has the same distance to the adjacent bases B_1 and B_1'. By triangle condition (3.1) there exists a base $C \in [B, B_1] \cap [B, B_1']$ adjacent to B_1 and B_1'. Since $B_1 \setminus \{j\} = B \cap B_1 \subseteq C \subseteq B \cup B_1' = (B_1 \cup B_2) \setminus \{j, l\}$, we conclude that $C = (B_1 \setminus \{j\}) \cup \{m\}$ for some $m \in B_2 \setminus \{l\}$. By (SEP) applied to $B_2, \emptyset$ and element $m \in B_2 \,\Delta\, \emptyset$, we conclude that there exists $m' \in B_2 \setminus \{m\}$ such that $D := \{m, m'\} \in \mathcal{B}_\varphi$. Since $d(C, \emptyset) = |C| = |B_1|$ and $d(C, D) = |(B_1 \setminus \{j\}) \cup \{m'\}| = |B_1|$, by (3.1) there exists a base $C' \in [C, \emptyset] \cap [C, D]$ adjacent to $\emptyset$ and D. Since $\{m\} = C \cap D \subseteq C' \subseteq C$, we deduce that $C' = \{e, m\}$ for some $e \in B_1 - \{j\}$. Then $e \in B_1 \subseteq I_1$ and $m \in B_2 \subseteq I \setminus I_1$, therefore the vertices e and m belong to distinct connected components of the graph Γ, contrary to the fact that $C' = \{e, m\}$ is a base of $\mathcal{B}_\varphi$. This final contradiction shows that $B' \cup B'' = B_1 \cup B_2$ must be a base, thus establishing that $\mathcal{B}_\varphi = \mathcal{B}_1 + \mathcal{B}_2$. $\square$

Proof of Proposition 2. Next suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two even Δ-matroids having the same (unlabelled) indecomposable basis graph G. For a vertex v of G, denote by $B_1(v)$ and $B_2(v)$ the bases of $\mathcal{B}_1$ and $\mathcal{B}_2$ represented by v. Pick some base-point b of G. Since the composition of twistings is a twisting and the class of even Δ-matroids is closed with respect to this operation, one can assume without loss of generality that $B_1(b) = B_2(b) = \emptyset$. One can easily see that $B_1(v) = \bigcup\{B_1(x)\colon x \in C_v\}$ and $B_2(v) = \bigcup\{B_2(x)\colon x \in C_v\}$ for each $v \nsim b$. The collections $\mathcal{B}_1' = \{B_1(x)\colon x \in N(b)\}$ and $\mathcal{B}_2' = \{B_2(x)\colon x \in N(b)\}$ can be viewed as edge-sets of two labelled connected graphs $\Gamma_1 = (I_1, F_1)$ and $\Gamma_2 = (I_2, F_2)$, both having $N(b)$ as a line graph. To Γ_1 and Γ_2 we will apply the following classical theorem of Whitney [25]: *If Γ_1 and Γ_2 are connected graphs and $L(\Gamma_1) \simeq L(\Gamma_2)$, then either $\Gamma_1 \simeq \Gamma_2$ or $\{\Gamma_1, \Gamma_2\} = \{K_3, K_{1,3}\}$. Further, for any isomorphism $\beta : L(\Gamma_1) \mapsto L(\Gamma_2)$ there exists a unique isomorphism $\alpha : \Gamma_1 \mapsto \Gamma_2$ inducing β, unless $\Gamma_1 \simeq \Gamma_2 \simeq K_{1,3} + e$, $\Gamma_1 \simeq \Gamma_2 \simeq K_4 - e$, or $\Gamma_1 \simeq \Gamma_2 \simeq K_4$.* As a consequence, if Γ_1 and Γ_2 do not form an exceptional pair, then any isomorphism between the line graphs $L(\Gamma_1) \simeq L(\Gamma_2) \simeq N(b)$ is induced by a point-by-point correspondence α of the ground sets I_1 and I_2. This bijection α maps the bases of $\mathcal{B}_1$ to bases of $\mathcal{B}_2$ (modulo twisting), and we are done.

If $\{\Gamma_1, \Gamma_2\} = \{K_3, K_{1,3}\}$, then $N(b) = K_3$. If G contains a fifth vertex a, then obviously $a \nsim b$, thus one can take a at distance 2 from b. Since b and a are contained in a common square, $N(b)$ is not a complete graph, a contradiction. Thus $G = K_4$ and $\mathcal{B}_1 = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and $\mathcal{B}_2 = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. (Every other even Δ-matroid with K_4 as a basis graph can be obtained from one of these two by a twisting.) Finally, suppose $\Gamma_1 \simeq \Gamma_2 \in \{K_{1,3} + e, K_4 - e, K_4\}$. Since $B_1(b) = B_2(b) = \emptyset$, $|B_1(x)| = |B_2(x)| = 2$ for any vertex x of $N(b)$, and $|I_1| = |I_2| = 4$, the basis graph G may contain only one extra-vertex a, encoded by $B_1(a) = I_1$ and $B_2(a) = I_2$. Hence G is an induced subgraph of the 4-octahedron, concluding the proof of Proposition 2. $\square$

6. Proof of Theorem 2

We start with the following characterization of even Δ-matroids (for matroids, an equivalence similar to (i) $\Leftrightarrow$ (iii) has been proven by A. Kelmans (unpublished); cf. [24]):

Proposition 3. *For $\mathcal{B} \subset 2^{[n]}$ the following conditions are equivalent*:

(i) *$\mathcal{B}$ is an even Δ-matroid*;
(ii) [21] *for every $A, B \in \mathcal{B}$ and $i \in A \,\Delta\, B$, there exists some $j \in A \,\Delta\, B$, $j \neq i$, such that $A \,\Delta\, \{i, j\} \in \mathcal{B}$ and $B \,\Delta\, \{i, j\} \in \mathcal{B}$*;
(iii) *for every $A, B \in \mathcal{B}$, there exists $i, j \in A \,\Delta\, B$, $j \neq i$, such that $A \,\Delta\, \{i, j\} \in \mathcal{B}$ and $B \,\Delta\, \{i, j\} \in \mathcal{B}$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is the main result of [21]. The implication (ii) $\Rightarrow$ (iii) is trivial. To show that (iii) $\Rightarrow$ (i), we apply Theorem 1. Obviously, all subsets of $\mathcal{B}$ have the same cardinality modulo 2. By induction on $d(A, B)$ one can easily show that $G(\mathcal{B})$ is an isometric (therefore connected) subgraph of the half-cube. It remains to establish (IC4). Let $d(A, B) = 2$, say $B = \emptyset$ and $A = \{1, 2, 3, 4\}$. Let $i = 1$ and $j = 2$ be the indices satisfying the exchange property (iii). Hence $B \,\Delta\, \{1, 2\} = \{1, 2\}$ and $A \,\Delta\, \{1, 2\} = \{3, 4\}$ are bases. Together with A and B they form a square of $G(\mathcal{B})$. $\square$

Let $\mathcal{B}$ be an even Δ-matroid. Since $G(\mathcal{B}) \subseteq \Pi_1(\mathcal{B})$ holds for any collection of even subsets, it remains to establish the converse inclusion. Suppose the segment connecting $\sigma(\emptyset)$ and $\sigma(B)$ is an edge of $\Pi_1(\mathcal{B})$, where $B = \{1, 2, 3, 4, \ldots, k\}$. By Proposition 3(iii) there exists $i, j \in B$, say $i = 1$, $j = 2$, such that both $\{1, 2\}$ and $B \setminus \{1, 2\}$ are bases. The point whose first k coordinates are equal to $\frac{1}{2}$ and the remaining coordinates are 0 is the middle of the segments between $\sigma(\emptyset), \sigma(B)$ and $\sigma(\{1, 2\}), \sigma(B \setminus \{1, 2\})$, respectively, contrary to our assumption.

Conversely, let $\mathcal{B}$ be a collection of sets of even cardinality such that $\Pi_1(\mathcal{B}) = G(\mathcal{B})$. Since $G(\mathcal{B})$ is a connected induced subgraph of $\frac{1}{2}H_n$, by Theorem 1(iv) it suffices to show that this graph satisfies (IC4). Pick a 2-interval $[A, B]$ of $G(\mathcal{B})$, say $B \,\Delta\, A = \{1, 2, 3, 4\}$. Let H be the 4-cube induced by the incidence vectors of the subsets of $[A, B]_{H_n}$. Suppose by way of contradiction that (A, B) does not contain two non-adjacent bases. Then the incidence vectors of all bases of $\mathcal{B}$ included in (A, B) lie in a common facet H' of H, say in a facet containing $\sigma(A)$. Applying a twisting, one can suppose $A = \emptyset$ and $B = \{1, 2, 3, 4\}$. Additionally suppose that H' is the 3-dimensional subcube of H induced by the incidence vectors of all subsets of B not containing the element 4. We claim that the segment joining $\sigma(A)$ and $\sigma(B)$ is an edge of $\Pi(\mathcal{B})$. Suppose not; then some interior point $p = (p_1, p_2, \ldots, p_n)$ of this segment can be written as a convex combination $\mu_1\sigma(B_1) + \cdots + \mu_k\sigma(B_k)$ of incidence vectors of some bases $B_1, \ldots, B_k$ (i.e., each μ_i is positive and $\sum_{i=1}^{k} \mu_i = 1$). Then $p_1 = p_2 = p_3 = p_4 = \lambda$ for some $0 < \lambda < 1$ and $p_j = 0$ for any other index j. This implies that no base B_i $(i = 1, \ldots, k)$, contains an element $j \notin \{1, 2, 3, 4\}$, whence all B_i are subsets of B. As $p \notin H'$, the base B must be one of B_i, say $B_1 = B$. Since $\sigma_4(B_i) = 0$ for $i > 1$, the equality $\mu_1 \cdot 1 + \sum_{i=2}^{k} \mu_i \cdot 0 = \lambda$ implies that $\mu_1 = \lambda$. Replacing μ_1 by λ in each of the equalities $\mu_1 \cdot 1 + \sum_{i=2}^{k} \mu_i\sigma_j(B_i) = \lambda$, $j = 1, 2, 3$, we obtain $\mu_i = 0$ for all B_i distinct from A and B. Hence the incidence vectors of the bases A, B define an edge of $\Pi_1(\mathcal{B})$, contrary to the initial assumption.

7. Proof of (ii) ⇒ (iii) of Theorem 1

Let $G = (V, E)$ be a graph satisfying the conditions (IC4) and (PC). To establish that every $N(v)$ $(v \in V)$ is a line graph provided it does not contain W_5 and W_6 as an induced subgraph, we will use the characterization of line graphs due to Beineke (cf. [1] or any other textbook on graph theory): *a graph H is a line graph if and only if none of the graphs W_5 and $F_1 - F_8$ of Fig.* 1 *is an induced subgraph of H*. From (IC4) we infer that induced F_2 or F_3 cannot occur in the neighborhood of some vertex of G. Each of remaining graphs requires special (and sometimes considerable) efforts. Assume by way of contradiction that $N(v)$ contains an induced subgraph $F = F_i$, $i = 4, \ldots, 8$, labelled as in Fig. 1.

Case $F = F_1$. Suppose $N(v)$ has a $K_{1,3}$ centered at w and with tips x_1, x_2, x_3. If there exists a vertex $u \in (x_1, x_2)$, $u \neq w$, not adjacent to one of v or w, say $u \nsim w$, then $u \sim v$ by (PC) so $d(u, x_3) \leqslant 2$, thus the square $x_1 u x_2 w$ and the vertex x_3 provide a counterexample to (PC). Hence, by (IC4) there exist 6 distinct vertices $a_1, a_2 \in (x_1, x_2)$, $b_1, b_2 \in (x_1, x_3)$, and $c_1, c_2 \in (x_2, x_3)$ which are all adjacent to v, w and such that $a_1 \nsim a_2$, $b_1 \nsim b_2$, $c_1 \nsim c_2$. Applying (PC) to these vertices and obtained squares, one can see that each a_i is adjacent to one b_j and one c_k and similar conclusions hold for b-vertices and c-vertices. The subgraph induced by new vertices is triangle-free: if, say, a_1, b_1, c_1 are pairwise adjacent then (c_1, x_1) contains a 4-clique induced by a_1, b_1, v, w, contrary to (IC4). Thus this subgraph is an induced 6-cycle, say $a_1 b_1 c_2 a_2 b_2 c_1$. Then we have a W_6 with center w in $N(v)$, a contradiction. This shows that F_1 is forbidden in $N(v)$, thus G does not contain induced propellers.

Case $F = F_4$. Since $x, y, v \in (a_1, b_1)$, by (IC4) there is $t \in (a_1, b_1)$ that is not adjacent to exactly one of x, y, v: without loss of generality suppose $t \nsim v$, $t \sim x, y$. By (PC) applied to

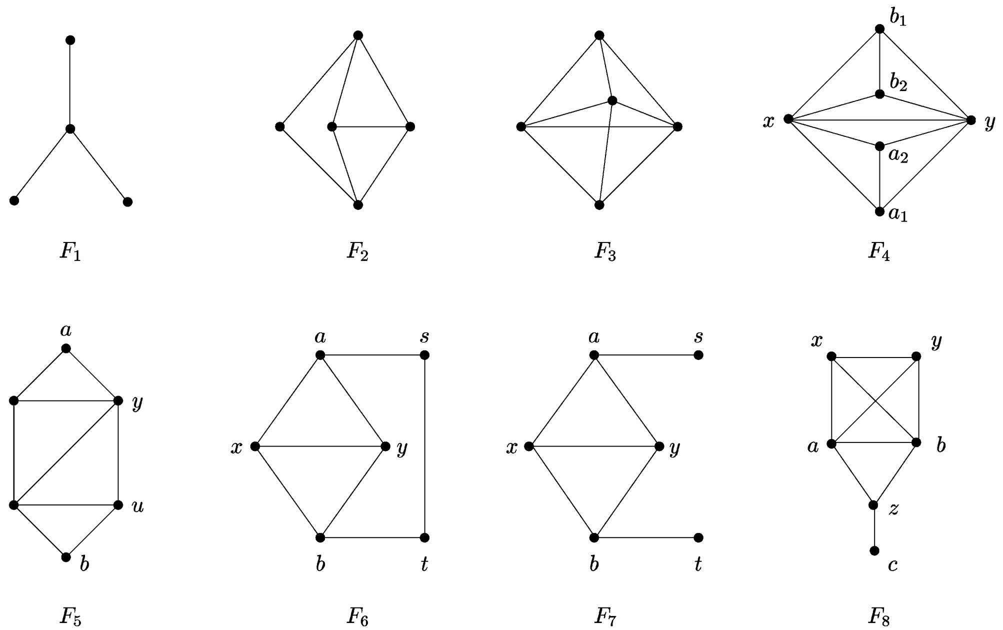

Fig. 1.

a_2 and the square a_1vb_1t, we have $t \nsim a_2$, and similarly $t \nsim b_2$. Now x, y, t, a_2, b_2 induce a propeller.

Case $F = F_5$. First suppose that $[a, b]$ contains two non-adjacent vertices w_1, w_2 distinct from v, hence $v \sim w_1, w_2$. To avoid propellers, each of the vertices x, y, z, u must be adjacent to one of w_1, w_2. Let $y \sim w_2$. Then $w_2 \in [y, b]$, therefore w_2 is adjacent to one of z, u. If $w_2 \sim z$, $w_2 \nsim u$, then $w_2 \sim x$ by (PC) applied to x and the square $yubw_2$. But then (z, a) contains a K_4 induced by x, y, v, w_2. So $w_2 \sim u$. Then $z \nsim w_2$, otherwise (y, b) would contain a K_4 induced by v, z, u, w_2. Finally, if $x \sim w_2$, then $[x, b]$ will be not convex because $z, w_2 \in [x, b]$ and $u \in [z, w_2] \setminus [x, b]$. Thus $w_2 \nsim x, z$. But then $N(v)$ contains a 5-wheel induced by a, w_2, u, z, x, y and centered at y. This shows that there exists a vertex $w \in (a, b)$ not adjacent to v. From (PC) we infer that w is not adjacent to any other current vertex.

Pick a new vertex $s \in (y, b)$. First suppose that $s \sim v$. Then $s \nsim z$ or $s \nsim u$. In any case, by (PC) we have $s \sim a, w$. Now, if $s \sim u$, then $s \nsim x, z$, and in $N(y)$ we obtain a W_5 induced by the vertices a, x, z, u, s, v and centered at v. Otherwise, if $s \sim z$, $s \nsim u$, by (PC) we conclude that $x \sim s$. Then however (z, a) will contain a K_4 induced by x, y, s, v. Therefore $s \nsim v$ and $s \sim z, u, w$, moreover, s is not adjacent to any other current vertex. Analogously, we can find a vertex $t \in (z, a)$ such that $t \nsim v$, $t \sim w, x, y$. Clearly $t \neq s$. By (PC) immediately follows that $t \sim s$. But then $N(y)$ contains a W_5 induced by x, v, t, u, s, z and centered at z.

Case $F = F_6$. Pick a new vertex $w \in (a, b)$. If $w \sim v$, then w is adjacent to exactly one of x, y, say $w \sim x$, $w \nsim y$. Then $w \sim s, t$ by (PC), and in $N(v)$ we get a 5-wheel induced by a, x, s, t, b, w and centered at w. Hence any new vertex of (a, b) is not adjacent to v. Pick a square $S_1 = yptp_1$ containing the vertices y, t. If p, p_1 are distinct from b, v, then $b, v \sim p, p_1$ by (IC4). By (PC), a is adjacent to one of p, p_1, say $a \sim p$. But then $p \in (a, b)$ and $p \sim v$, contrary to our previous conclusion. Thus one of p, p_1, say p_1, coincides with b or v.

(a) $p_1 = b$ and $p \sim v$. Then (PC) implies that $p \nsim x, w$ and $p \sim a, s$.
(b) $p_1 = v$ and $p \sim b$. Then (PC) implies that $p \nsim a, s, x$ and $p \sim w$.

Analogously, taking a square $S_2 = yqsq_1$ containing the vertices y, s, one may assume that q_1 coincides with a or v, leading to conclusions similar to (a) and (b) (by symmetry, one can define also the squares S_3 and S_4 for the pairs x, t and x, s, respectively). If the squares S_1, S_2 are both in position (b), then $p \sim q$ by (PC) and $N(w)$ will contain a W_5 induced by a, x, b, p, q, y and centered at y. Now, suppose that S_1 is in position (a) and S_2 is in position (b). Again $p \sim q$, because $p, q, v \in [y, s]$ and $q \nsim v$. Then the vertices a, q, p, v, x, b all belong to $N(y)$ and induce a forbidden F_5. Finally suppose that all four squares S_1, S_2, S_3, S_4 are in position (a). Let $S_3 = xp'tb$ and $S_4 = xq'sa$. Then $p, p' \sim a, s, v$ and $q, q' \sim b, t, v$, while $p, p' \nsim b, w$ and $q, q' \nsim a, w$. Additionally, $p, q \nsim x$ and $p', q' \nsim y$. From (PC) we conclude that $p \sim q$ and $p' \sim q'$. Applying (IC4) to the intervals $[b, s]$ and $[a, t]$, we obtain that $p \nsim p'$ and $q \nsim q'$. But then the vertex w and each of the squares $aptp'$ and $bq'sq$ provide a counterexample to (PC).

Case $F = F_7$. Pick a vertex $w \in (s, t)$. If $w \nsim v$, then applying (PC) to the square $svtw$ and each of the vertices x and y, we conclude that $d(x, w) = d(y, w) = 3$. Since $a, b \in [x, w]$ and $y \in [a, b] \setminus [x, w]$, we get a contradiction with the convexity of $[x, w]$. Thus there exist two non-adjacent vertices $w_1, w_2 \in (a, b)$ both adjacent to v. By (PC) each of a, b is adjacent to one of w_1, w_2. If both a, b are adjacent to the same vertex, say to w_1, then $x, y, w_1 \in (a, b)$. Thus w_1 must be adjacent to one of x, y, say $w_1 \sim x$. Then the square sw_1tw_2 and the vertex x contradict (PC). Thus let $a \sim w_1, a \nsim w_2$ and $b \sim w_2, b \nsim w_1$. Then however $N(v)$ contains a forbidden F_6 induced by a, x, y, b, t, w_1.

Case $F = F_8$. Pick $s \in (b, c), s \neq z$. First, let $s \sim v$. If $s \nsim z$, from (PC) we obtain $s \sim x, y$ and $s \nsim a$. Then (a, s) contains a K_4 induced by v, b, x, y, a contradiction. So $s \sim z$ and then $s \nsim a$, otherwise (b, c) contains K_4 induced by v, a, z, s. Now since (s, a) does not have a K_4 induced by b, v, x, y, we see that s is not adjacent to one of the vertices x, y, say $s \nsim x$. But then $N(v)$ contains an F_5 induced by the vertices x, a, b, z, s, c. Thus any new vertex of $[b, c]$ or $[a, c]$ is not adjacent to v. Take two squares $S' = bvcp$ and $S'' = avcq$. From (PC) one infer that the vertices p and q are distinct but adjacent and both are adjacent to z. The same condition implies that $p \nsim a, x, y$ and $q \nsim b, x, y$.

Applying (IC4) for $[y, c]$ and (PC) for q and $[y, c]$, one conclude that there is a vertex $s_1 \in (y, c)$ adjacent to q. If $s_1 \sim v$, then $s_1 \in [v, q] \subseteq [a, c]$, contrary to what has been concluded above. Hence $s_1 \nsim v$. By (PC) one has $p \sim s_1$. Now, if $b \sim s_1$, then $s_1 \in [b, c]$, implying that s_1 and v are adjacent. Thus $b \nsim s_1$. Applying (PC) to the square $pbys_1$ and the vertices a, x, z, one conclude that $s_1 \nsim a, x, z$. Applying (IC4) for $[z, s_1]$ and (PC), one conclude that there exists a vertex $t_1 \in (z, s_1)$, $t_1 \sim y$, adjacent to two of the vertices p, c, q. Since $t_1 \in [z, y]$, t_1 is also adjacent to exactly two of the vertices a, v, b. Now, if $t_1 \sim c$, then $t_1 \in [c, y]$, therefore $t_1 \sim v$. As t_1 is adjacent to a or b, we have $t_1 \in [a, c] \cup [b, c]$, which is impossible by what has been shown above. So, assume $t_1 \nsim c$. From (PC) applied to v and the square t_1zcs_1 we infer that $t_1 \nsim v$. Hence $t_1 \sim p, q, a, b$, on the other hand, $t_1 \nsim x$ by (PC) applied to x and the square t_1zcs_1.

Analogously, there exist vertices s_2 and t_2, such that $s_2 \sim x, p, q, c$, $t_2 \sim x, s_2, a, b, z, p, q$ and $t_2 \nsim y, c, v$. If $t_1 \nsim t_2$, then applying (PC) to c and the square t_1at_2p we infer that c must be adjacent to t_1 or t_2. Since this is impossible, we deduce that $t_1 \nsim t_2$. But then $N(z)$ contains a forbidden F_6 induced by b, t_1, t_2, q, v, c. This contradiction completes the proof of the implication (ii) $\Rightarrow$ (iii) of Theorem 1.

The interval condition (IC4) and the positioning condition (PC) solely do not characterize basis graphs of even Δ-matroids. The following graphs H_1 and H_2 obey (IC4) and (PC), nevertheless the neighborhoods of several vertices contain induced 5- or 6-wheels. H_1 is the graph induced by 11 vertices defined in the analysis of case $F = F_1$. We noticed already that H_1 satisfies (IC4) and (PC). Notice also that $N(v)$ contains a 6-wheel induced by the 6-cycle $a_1b_1c_2a_2b_2c_1$ and centered at w. On the other hand, one can directly check that W_5 does not occur in the neighborhoods of vertices of H_1. The graph H_2 contains 14 vertices $x_1, \dots, x_4, y_1, y_2, u_1, \dots, u_4, v_1, \dots, v_4$, where the x-vertices induce a square, the u- and the v-vertices induce two K_4, and, additionally, u_i and v_j are adjacent if and only if $i = j$. Furthermore, $y_1 \sim y_2$ and both y_1, y_2 are adjacent to all x-vertices. Additionally, y_1 is adjacent to all u-vertices and y_2 is adjacent to all v-vertices. Finally, the vertices u_i, v_i are adjacent to the vertices of the edge x_i, x_{i+1} (here indices are taken modulo 4). Using the symmetry and the fact that H_2 has diameter 2, one can show that H_2 satisfies (IC4) and (PC). On the other hand, $N(y_2)$ contains a 5-wheel induced by $v_1, v_2, x_3, y_1, x_1, x_2$ and centered at x_2. Finally notice that W_6 does not occur in the neighborhoods of vertices of the graph H_2.

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