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NOTE

Bridged Graphs Are Cop-Win Graphs: An Algorithmic Proof

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A graph is *bridged* if it contains no isometric cycles of length greater than three. Anstee and Farber established that bridged graphs are cop-win graphs. According to Nowakowski and Winkler and Quilliot, a graph is a cop-win graph if and only if its vertices admit a linear ordering $v_1, v_2, ..., v_n$ such that every vertex v_i, $i > 1$, is dominated by some neighbour v_j, $j < i$, i.e., every vertex v_k, $k < i$, adjacent to v_i is adjacent to v_j, too. We present an alternative proof of the result of Anstee and Farber, which allows us to find such an ordering in time linear in the number of edges. Namely, we show that every ordering of the vertices of a bridged graph produced by the breadth-first search is a "cop-win ordering."

1. INTRODUCTION

An induced subgraph H of a graph G is *isometric* if the distance between any pair of vertices in H is the same as that in G. A graph is *bridged* if it contains no isometric cycles of length greater than three. As was shown in Soltan and Chepoi [6] and Farber and Jamison [3], bridged graphs are exactly those graphs whose metric convexity enjoys an important property of Euclidean convexity, that the neighbourhoods of convex sets are convex.

Nowakowski and Winkler [4] and Quilliot [5] considered a game of a cop and a robber on the vertices of a graph. The players begin the game by selecting their initial positions in the graph (the cop must choose his vertex first). They then move alternatively, according to the following rule: a player at vertex v can either remain at v or move to any neighbour of v. The cop wins when the cop and robber occupy the same vertex. The graphs

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with winning strategy for the cop were dubbed "cop-win graphs." Nowakowski and Winkler [4] and Quilliot [5] gave a complete description of these graphs:

> *A graph is cop-win if and only if its vertices can be linearly ordered, $v_1, v_2, ..., v_n$, so that, for each v_i, $i>1$, there is a neighbour v_j, $j<i$, of v_i, such that every vertex v_k, $k<i$, adjacent to v_i is also adjacent to v_j.*

We will call such an ordering of the vertices of G a *cop-win ordering*. That bridged graphs are cop–win graphs has been established by Anstee and Farber [1] in 1988:

Every bridged graph is a cop-win graph.

Their proof uses a variety of properties previously proved for bridged graphs in Farber [2], Farber and Jamison [3], and Soltan and Chepoi [6]. The purpose of this note is to prove that any ordering of the vertices of a bridged graph produced by the breadth-first search is a cop-win ordering.

2. THE RESULT

Recall that an induced subgraph (or a subset of vertices) H is called *convex* if H includes every shortest path with end-vertices in H. For a subset K of the vertices of G, we denote by $N[K]$ the *closed neighbourhood* of K, i.e., the set of vertices which are equal or adjacent to some vertex in K. The following result provides a convexity characterization of bridged graphs.

THEOREM A [3,6]. *A graph G is bridged if and only if for every convex set K the set $N[K]$ is convex.*

For an integer $k \geqslant 1$ and a subset K let $N_k[K]$ be the *k-iterated neighbourhood* of K, i.e.

$$N_k[K] = \{v \in V: d(v, u) \leqslant k \text{ for some } u \in K\}.$$

Since $N_k[K] = N[N_{k-1}[K]]$, it follows from Theorem A that in bridged graphs all iterated neighbourhoods of convex sets are convex.

In a *breadth-first search* (BFS) the vertices of a graph G with n vertices are numbered from 1 to n in increasing order. We number with 1 a vertex u and put it on an initially empty queue of vertices. We repeatedly remove the vertex v at the head of the queue and consequently number and place onto the queue all still unnumbered neighbours of v. BFS constructs a rooted spanning tree T of G with the vertex u as a root. Then a vertex v is the *father* in T of exactly those neighbours in G which are included in the queue when v is removed. The procedure is executed once for each vertex, so the total complexity of its implementation is $O(|V|+|E|)$.

For a given linear ordering $v_1, v_2, ..., v_n$, of the vertices of G we let G_i denote the subgraph induced by $\{v_1, v_2, ..., v_i\}$. For arbitrary vertices x and y we put $x \prec y$ whenever $x=v_i$, $y=v_j$ and $i<j$.

THEOREM. *Any ordering $v_1, ..., v_n$, of the vertices of a bridged graph G produced by the breadth-first search is a cop-win ordering.*

Proof. Suppose that the breadth-first search has u as a starting point, i.e. $v_1=u$. First we will verify the following assertion.

CLAIM. *Let v and w be two adjacent vertices of G which are equidistant to u. If x and y are the fathers of v and w, respectively, then x and y either coincide or are adjacent. In addition, if $v \prec w$, then y is adjacent to v.*

Proof of the Claim. We proceed by induction on the distance $k=d(u, v)=d(u, w)$. If $k=1$, then $x=u=y$, and we are done. So, let $k>1$. Suppose by way of contradiction that $d(x, y)>1$. Since x and y are at distance $k-1$ to u, the convexity of the set $N_{k-1}[u]$ implies that $d(x, y)=2$. Moreover, from the same fact follows that the path $xvwy$ must be induced. Let z be a common neighbour of x and y. Since G does not contain induced 5-cycles or 4-cycles, the vertex z is adjacent to both v and w. Thus, $d(z, u)=k-1$. Consider the fathers p, t and q of the vertices x, z and y, respectively. By the induction hypothesis $d(p, t) \leqslant 1$ and $d(t, q) \leqslant 1$. In addition, t is adjacent to both x and y, because $x \prec z \succ y$ by BFS. Then, however, the vertices t, x, v, w, y induce a 5-cycle, which is impossible. This shows that $d(x, y) \leqslant 1$. If $v \prec w$, then according to BFS $x \prec y$, too. If y and v were non-adjacent, then x and w are adjacent, contrary to the fact that y is the father of w. ▮

In order to complete the proof of the theorem, we show by induction on i that the vertex $w=v_i$ is dominated in G_i by its father y, i.e. y is adjacent to any neighbour v of w in G_i. Let $k=d(u, w)$. If $d(u, v)=k$, then y and v must be adjacent according to the claim. Otherwise, if $d(u, v)=d(u, y)=k-1$, then $v, y \in N_{k-1}[u]$ and $w \notin N_{k-1}[u]$. And again, y and v must be adjacent, because the set $N_{k-1}(u)$ is convex. This finishes the proof of the theorem. ▮

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REFERENCES

1. R. P. Anstee and M. Farber, On bridged graphs and cop–win graphs, *J. Combin. Theory Ser. B* **44** (1988), 22–28.
2. M. Farber, Bridged graphs and geodesic convexity, *Discrete Math.* **66** (1987), 249–257.
3. M. Farber and R. E. Jamison, On local convexity in graphs, *Discrete Math.* **66** (1987), 231–247.
4. R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43** (1983), 235–239.
5. A. Quilliot, "Homomorphismes, points fixes, rétractions et jeux de poursuite dans les graphes, les ensembles ordonnés et les espaces métriques," Thèse d'État, Université de Paris VI, 1983.
6. V. Soltan and V. Chepoi, Conditions for invariance of set diameters under *d*-convexification in a graph, *Cybernetics* **19** (1983), 750–756.