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Approximating hitting sets of axis-parallel rectangles intersecting a monotone curve

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ABSTRACT

In this note, we present a simple combinatorial factor 6 algorithm for approximating the minimum hitting set of a family $\mathcal{R} = \{R_1, \ldots, R_n\}$ of axis-parallel rectangles in the plane such that there exists an axis-monotone curve γ that intersects each rectangle in the family. The quality of the hitting set is shown by comparing it to the size of a packing (set of pairwise non-intersecting rectangles) that is constructed along, hence, we also obtain a factor 6 approximation for the maximum packing of $\mathcal{R}$.

In cases where the axis-monotone curve γ intersects the same side (e.g. the bottom side) of each rectangle in the family the approximation factor for hitting set and packing is 3.

1. Introduction

Let $\mathcal{R} = \{R_1, \ldots, R_n\}$ be a family of axis-parallel rectangles of $\mathbb{R}^2$. A set of points $T \subset \mathbb{R}^2$ is said to be a *transversal* or a *hitting or piercing set* of $\mathcal{R}$ if $T \cap R_i \neq \emptyset$ for any $R_i \in \mathcal{R}$. The *transversal number* $\tau(\mathcal{R})$ is the minimum size of a hitting set of $\mathcal{R}$. The *packing number* $\nu(\mathcal{R})$ is the maximum number of pairwise disjoint rectangles of $\mathcal{R}$. In terms of the intersection graph, $G_{\mathcal{R}}$, of the family of rectangles the packing number is the independence number $\alpha(G_{\mathcal{R}})$ and due to the Helly property of axis-parallel rectangles the transversal number equals the clique covering number $\theta(G_{\mathcal{R}})$. Since $\alpha(G_{\mathcal{R}}) \leqslant \theta(G_{\mathcal{R}})$ we also have $\nu(\mathcal{R}) \leqslant \tau(\mathcal{R})$ for every family $\mathcal{R}$.

Computing, approximating, and relating $\tau(\mathcal{R})$ and $\nu(\mathcal{R})$ is both an algorithmic and combinatorial question with numerous applications. In 1965, Wegner [19] asked if it is always true that $\tau(\mathcal{R}) \leqslant 2\nu(\mathcal{R}) - 1$ and Gyárfás and Lehel [13] relaxed this question by asking if $\tau(\mathcal{R}) \leqslant c\nu(\mathcal{R})$ for a universal constant c not depending on $\mathcal{R}$. In [13] they also noticed that $\tau(\mathcal{R}) \leqslant \nu^2(\mathcal{R})$. Károlyi [15] proved that $\tau(\mathcal{R}) \leqslant \nu(\mathcal{R})\lfloor \log \tau(\mathcal{R}) \rfloor + 2$. A simpler proof of this result was given by Fon-Der-Flaass and Kostochka [11]; they also construct a family $\mathcal{R}$ consisting of 23 rectangles such that $\tau(\mathcal{R}) \geqslant \frac{5}{3}\nu(\mathcal{R})$. Nielsen [17] showed that $\tau(\mathcal{R}) \leqslant 2\nu(\mathcal{R})$ if $\mathcal{R}$ consists of unit squares and Ahlswede and Karapetyan [3] announced that $\tau(\mathcal{R}) \leqslant 4\nu(\mathcal{R})$ if $\mathcal{R}$ is a family of squares.

Let P_b and P_r be two finite sets of points in the plane and let $\mathcal{R}$ be the family of all rectangles with bottom left corner in P_b (blue) and top right corner in P_r (red). Soto and Telha [18] showed that in this case $\tau(\mathcal{R}) = \nu(\mathcal{R})$, moreover optimal transversals and packings can be computed efficiently. In general the problems of computing the transversal and packing numbers of a family of axis-parallel rectangles are NP-hard. Hardness has been proven even for the case when all rectangles are unit squares (Fowler et al. [10]).

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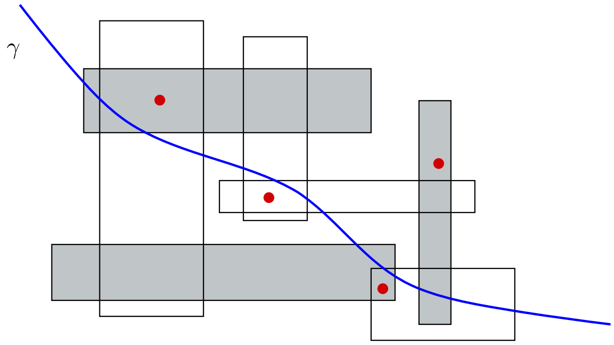

Fig. 1. A family $\mathcal{R}$ of seven rectangles intersected by an axis-monotone curve γ. The intersection graph $G_{\mathcal{R}}$ is a 7-cycle, hence, $\tau(\mathcal{R}) = 4$ and $\nu(\mathcal{R}) = 3$.

Hochbaum and Maass [14] presented a PTAS for approximating $\tau(\mathcal{R})$ for unit squares and Chan [7] provided a PTAS for arbitrary axis-parallel squares. Hitting sets have been studied intensely in the context of range spaces and ϵ-nets. Aronov, Ezra, and Sharir [2] proved the existence of $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$-nets for families of axis-parallel rectangles, which, combined with a result of Brönnimann and Goodrich [4], leads to a factor $O(\log\log\tau(\mathcal{R}))$ approximation algorithm for the transversal number $\tau(\mathcal{R})$. Mustafa and Ray [16] show that the approach yields a PTAS for families of rectangles of unit height.

Agarwal and Mustafa [1] presented a constant factor approximation of $\nu(\mathcal{R})$ when the rectangles of $\mathcal{R}$ are pseudodiscs, i.e., the intersection of the boundaries of any two rectangles consists of at most two points. More recently, Chan and Har-Peled [8] extended the approach of [1] to arbitrary pseudodiscs and presented a PTAS for approximating $\nu(\mathcal{R})$. Chan and Har-Peled [8] noticed that in this case $\nu(\mathcal{R}) = O(\tau(\mathcal{R}))$ holds. Chalermsook and Chuzhoy [6] described an $O(\log\log n)$ approximation algorithm for approximating $\nu(\mathcal{R})$ for a set $\mathcal{R}$ of n rectangles.

In this note, we present a factor 6 approximation algorithm for $\tau(\mathcal{R})$ and a corresponding factor 6 approximation for $\nu(\mathcal{R})$ for families $\mathcal{R}$ of axis-parallel rectangles intersected by an axis-monotone curve γ. The approximation factors are obtained by constructing a hitting set T and a packing $\mathcal{P}$ such that $|T| \leqslant 6|\mathcal{P}|$, whence $\tau(\mathcal{R}) \leqslant |T| \leqslant 6|\mathcal{P}| \leqslant 6\nu(\mathcal{R})$.

An *axis-monotone curve* is an unbounded Jordan curve γ such that the intersection of γ with each horizontal or vertical line is a single point or an interval. An axis-monotone curve γ separates the plane into two halves H'_γ and H''_γ. Axis-monotone curves come in two types: they either go from north-west to south-east or from south-west to north-east. More formally, if γ is axis-monotone and $p, q, r \in \gamma$ with $p_x < q_x < r_x$ then either $p_y > q_y > r_y$ or $p_y < q_y < r_y$. In our exposition we assume that axis-monotone curves are of the first type, i.e., from north-west to south-east.

We say that a family of axis-parallel rectangles $\mathcal{R}$ is *separable* if there exists an axis-monotone curve γ intersecting all rectangles in $\mathcal{R}$. Since γ is assumed to go from north-west to south-east the top right corner and the bottom left corner of each rectangle belong to H'_γ and H''_γ respectively. One can easily show by examples (e.g. Fig. 1) that for separated families of rectangles the graph $G_{\mathcal{R}}$ may contain odd induced cycles, therefore it is not perfect and in general we have $\tau(\mathcal{R}) > \nu(\mathcal{R})$).

Here is the main result of this note:

Theorem 1. *If a family $\mathcal{R}$ of n axis-parallel rectangles is separable, then $\tau(\mathcal{R}) \leqslant 6\nu(\mathcal{R})$. Testing if $\mathcal{R}$ is separable and constructing a hitting set T of size at most $6\tau(\mathcal{R})$ and a packing P of size at least $\nu(\mathcal{R})/6$ can be done in $O(n\log n)$ time.*

For the proof we first partition $\mathcal{R}$ into those rectangles where γ intersects the bottom side and those where γ intersects the right side. For each of the two classes of the partition we construct a hitting set and a packing whose size differs at most by a factor of 3. Technically speaking we show:

Theorem 2. *If a family $\mathcal{R}$ of n axis-parallel rectangles is separable and there is an axis-monotone curve γ that intersects all the rectangles of $\mathcal{R}$ on the right side, then $\tau(\mathcal{R}) \leqslant 3\nu(\mathcal{R})$. Testing the property and constructing a hitting set T of size at most $3\tau(\mathcal{R})$ and a packing P of size at least $\nu(\mathcal{R})/3$ can be done in $O(n\log n)$ time.*

The case where a straight line ℓ exists such that each rectangle of $\mathcal{R}$ has a corner on ℓ and is contained in a halfplane H'_ℓ has recently been studied by Catanzaro et al. [5]. In this case $\tau(\mathcal{R}) \leqslant 2\nu(\mathcal{R})$. The construction in [5] is closely related to our algorithm and carries over to the case where line ℓ is replaced by an axis-monotone curve γ.

An easy consequence of Theorem 1 together with Lemma 1 is that any family $\mathcal{R}$ of rectangles that can be stabbed by k lines has $\tau(\mathcal{R}) \leqslant 6k\nu(\mathcal{R})$.

2. Preliminary results

We begin with a simple lemma that allows us to decompose packing and hitting problems.

Lemma 1. *Suppose that a family of sets $\mathcal{F}$ is partitioned into m subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and that for each $\mathcal{F}_i$ there exists a polynomial algorithm that computes a hitting set T_i and a packing P_i of $\mathcal{F}_i$ such that $|T_i| \leqslant k_i|P_i|$. Then*

(a) $\bigcup_{i=1}^m T_i$ *is a hitting set of size at most* $(k_1 + \cdots + k_m)\tau(\mathcal{F})$.
(b) *The largest of the sets* P_i *is a packing of size at least* $\nu(\mathcal{F})/(k_1 + \cdots + k_m)$.

This leads to a factor $k_1 + \cdots + k_m$ approximation algorithms for the minimum hitting set and the maximum packing problems for $\mathcal{F}$. Moreover,

(c) $\tau(\mathcal{F}) \leqslant (k_1 + \cdots + k_m)\nu(\mathcal{F})$.

Proof. An optimal hitting set for $\mathcal{F}_i$ has size at least $|P_i|$, i.e., $|P_i| \leqslant \tau(\mathcal{F}_i)$. Therefore, $|T_i| \leqslant k_i\tau(\mathcal{F}_i) \leqslant k_i\tau(\mathcal{F})$ and $|\bigcup_{i=1}^m T_i| \leqslant \sum_{i=1}^m |T_i| \leqslant \sum_{i=1}^m k_i\tau(\mathcal{F}) = (k_1 + \cdots + k_m)\tau(\mathcal{F})$.

Since $\bigcup_{i=1}^m T_i$ is a hitting set, we obtain $\sum_{i=1}^m k_i|P_i| \geqslant \sum_{i=1}^m |T_i| \geqslant |\bigcup_{i=1}^m T_i| \geqslant \nu(\mathcal{F})$. It follows that if P_{i_0} is the largest of the sets P_i, then $(k_1 + \cdots + k_m)|P_{i_0}| \geqslant \nu(\mathcal{F})$.

For the final part (c) note that $\tau(\mathcal{F}) \leqslant |\bigcup_{i=1}^m T_i| \leqslant \sum_{i=1}^m |T_i| \leqslant \sum_{i=1}^m k_i|P_i| \leqslant (k_1+\cdots+k_m)|P_{i_0}| \leqslant (k_1+\cdots+k_m)\nu(\mathcal{F})$. □

A family of axis-parallel rectangles is said to be *linearly separable* if there exists an axis-monotone Jordan curve γ such that for each rectangle $R \in \mathcal{R}$ the intersection $R \cap \gamma$ is a non-empty subcurve of γ and for any $R', R'' \in \mathcal{R}$ we have $R' \cap R'' \neq \emptyset$ if and only if $R' \cap R'' \cap \gamma \neq \emptyset$.

Lemma 2. *If $\mathcal{R}$ is linearly separable, then $\tau(\mathcal{R}) = \nu(\mathcal{R})$.*

Proof. Let $\mathcal{I}_\gamma := \{R \cap \gamma\colon\ R \in \mathcal{R}\}$. First notice that since the separating curve γ is homeomorphic to the real line $\mathbb{R}$, up to this homeomorphism, $\mathcal{I}_\gamma$ can be viewed as a family of intervals in $\mathbb{R}$. Consider the interval graph G defined by $\mathcal{I}_\gamma$ and note that $\nu(\mathcal{I}_\gamma) = \alpha(G)$ and due to the Helly property of intervals $\tau(\mathcal{I}_\gamma) = \theta(G)$. Since interval graphs are perfect (cf. [12]) we obtain $\tau(\mathcal{I}_\gamma) = \nu(\mathcal{I}_\gamma)$. Thus, it suffices to show that $\tau(\mathcal{R}) = \tau(\mathcal{I}_\gamma)$ and $\nu(\mathcal{R}) = \nu(\mathcal{I}_\gamma)$. The second equality is obvious because γ is a linear separating curve, i.e., two rectangles of $\mathcal{R}$ are disjoint if and only if their intersections with γ are disjoint. From the definition of $\mathcal{I}_\gamma$ it follows that any hitting set of $\mathcal{I}_\gamma$ is also a hitting set of $\mathcal{R}$, hence, $\tau(\mathcal{R}) \leqslant \tau(\mathcal{I}_\gamma)$. Together with $\tau(\mathcal{I}_\gamma) = \nu(\mathcal{I}_\gamma) = \nu(\mathcal{R}) \leqslant \tau(\mathcal{R})$ this yields $\tau(\mathcal{R}) = \tau(\mathcal{I}_\gamma)$. □

A family $\mathcal{R}$ of axis-parallel rectangles is *cross separable* if there exists an axis-monotone Jordan curve γ such that either γ intersects the left and the right side of all rectangles R of $\mathcal{R}$ or γ intersects the top and the bottom side of all rectangles R of $\mathcal{R}$. In the first case we say that $\mathcal{R}$ is $\|$-*cross separable* while in the second case γ is $=$-*cross separable*.

Lemma 3. *If $\mathcal{R}$ is cross separable, then $\mathcal{R}$ is linearly separable.*

Proof. Suppose without loss of generality that $\mathcal{R}$ is $\|$-cross separated by γ. Consider the vertical projection π from $\mathbb{R}^2$ to $\mathbb{R}$. Since γ intersects the left and the right side of each rectangle R we have $\pi(R) = \pi(R \cap \gamma)$ for all R in $\mathcal{R}$. Now, if $R' \cap R'' \neq \emptyset$, then there is a point p in $\pi(R') \cap \pi(R'') = \pi(R' \cap R'') \neq \emptyset$. Axis-monotonicity of γ implies that $s = \pi^{-1}(p) \cap \gamma$ is a point or a vertical segment. Since $p \in \pi(R' \cap \gamma)$ the intersection of s and R' is non-empty and since γ only intersects the left and the right side of R' this implies that $s \subset R'$. Similarly, $s \subset R''$, hence, $R' \cap R'' \cap \gamma \neq \emptyset$ as required. □

3. The algorithm and its analysis

Let $\mathcal{R}$ be a separable family of axis-parallel rectangles and let γ be an axis-monotone curve intersecting all rectangles in $\mathcal{R}$. Recall that we assume that γ goes from north-west to south-east. It follows that γ intersects either the top or the left side and either the bottom or the right side of each rectangle in $\mathcal{R}$. Partition $\mathcal{R}$ into subfamilies $\mathcal{R}_b$, $\mathcal{R}_r$ where $\mathcal{R}_b$ consists of all rectangles in $\mathcal{R}$ whose bottom side is intersected by γ and $\mathcal{R}_r = \mathcal{R} \setminus \mathcal{R}_b$, i.e., γ intersects the right side of all $R \in \mathcal{R}_r$.

Next we describe a simple algorithm which construct a hitting set for the family $\mathcal{R}_b$. (For $\mathcal{R}_r$ we can use the same algorithm after reflecting the plane with respect to the line $y = -x$.) The idea is to partition the rectangles of $\mathcal{R}_b$ into two subfamilies $\mathcal{R}'$ and $\mathcal{R}''$. For the first family $\mathcal{R}'$, we construct a hitting set $T' \cup T^0$ and a packing $\mathcal{P}' \subset \mathcal{R}'$ such that $|T'| = |\mathcal{P}'|$ and $|T^0| \leqslant |T'|$. For the second family $\mathcal{R}''$ in the partition we can prove that it is $\|$-cross separable by the axis-monotone curve μ which is the upper zigzag of the points of T', thus by Lemmata 2 and 3 we conclude that $\tau(\mathcal{R}'') = \nu(\mathcal{R}'')$ and that an optimal hitting set and an optimal packing for $\mathcal{R}''$ can be computed efficiently.

Recall that a point $p = (p_x, p_y)$ is said to *dominate* a point $q = (q_x, q_y)$ if $q_x \leqslant p_x$ and $q_y \leqslant p_y$. For a finite set $X \subset \mathbb{R}^2$ let X_0 be the set of all points of X that are not dominated by any other point in X. The set X_0 is just the set of maxima of the dominance order on X. The *upper zigzag* $\mu(X)$ of X is the axis-monotone staircase passing through all points of X_0. Equivalently, the upper zigzag $\mu(X)$ is the boundary ∂U of the union $U = \bigcup_{p \in S} Q_p$ of the closed quadrants

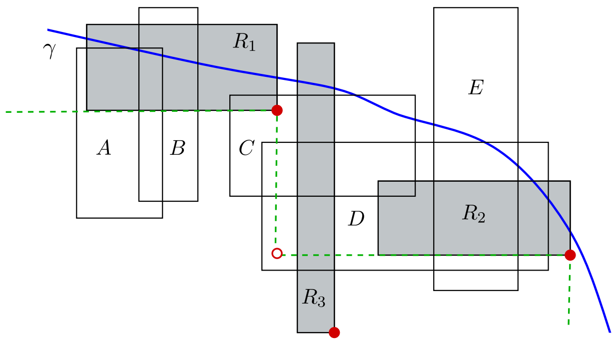

Fig. 2. R_1 is the first rectangle for $\mathcal{P}'$; since A, B, and C intersect R_1 they are moved to $\mathcal{R}''$. When R_2 moves to $\mathcal{P}'$, rectangles D and E are moved to $\mathcal{R}''$. Finally $\mathcal{P}' = \{R_1, R_2, R_3\}$. Rectangles C and D are moved from $\mathcal{R}''$ to $\mathcal{R}'$ because they contain a corner of the dashed zigzag. The final partition is $\mathcal{R}' = \{C, D\}$, $\mathcal{R}'' = \{A, B, E\}$.

$Q_p = \{q = (q_x, q_y) \in \mathbb{R}^2\colon\ q_x \leqslant p_x$ and $q_y \leqslant p_y\}$ consisting of all points of $\mathbb{R}^2$ dominated by $p = (p_x, p_y)$. Notice that $\mu(X)$ is an axis-monotone polygonal line whose convex corners are the points of X_0. (The *lower zigzag* $\lambda(S)$ of S can be defined analogously: in this case, the domination is considered with respect to the total order $\geqslant$ instead of $\leqslant$.)

A run of the following algorithm is exemplified in Fig. 2.

Algorithm HittingSet($\mathcal{R}_b$)

Input: The family $\mathcal{R}_b$.
Output: A partition of $\mathcal{R}_b$ into two families $\mathcal{R}'$, $\mathcal{R}''$ together with
a hitting set $T' \cup T^0$ and a packing $\mathcal{P}'$ of $\mathcal{R}'$ and
a hitting set T'' and a packing $\mathcal{P}''$ of $\mathcal{R}''$.
Initialization: $T' \leftarrow \emptyset$, $T^0 \leftarrow \emptyset$, and $\mathcal{P}' \leftarrow \emptyset$

1. **while** $\mathcal{R}_b \neq \emptyset$ **do**
2. Pick any R of $\mathcal{R}_b$ with a highest bottom side and let c_R be the bottom right corner of R.
3. Set $\mathcal{P}' \leftarrow \mathcal{P}' \cup \{R\}$ and $T' \leftarrow T' \cup \{c_R\}$.
4. Remove from $\mathcal{R}_b$ all rectangles R'' that intersect the rectangle R and insert them into $\mathcal{R}''$.
5. **endwhile**
6. Let $\mu(T')$ be the upper zigzag of T' and let T^0 be the set of all concave corners of $\mu(T')$.
7. Remove from $\mathcal{R}''$ all rectangles R'' such that $R'' \cap (T' \cup T^0) \neq \emptyset$ and insert them into $\mathcal{R}'$.
8. Compute a hitting set T'' and a packing $\mathcal{P}''$ of the cross-separable family $\mathcal{R}''$.
9. Return the subfamilies $\mathcal{R}'$, $\mathcal{R}''$, $\mathcal{P}'$, and $\mathcal{P}''$ of $\mathcal{R}_b$ and the point sets $T' \cup T^0$ and T''.

We begin the analysis of the algorithm by looking at the family $\mathcal{R}'$.

Lemma 4. *The set $T' \cup T^0$ is a hitting set and $\mathcal{P}'$ is a packing of $\mathcal{R}'$. The sizes of the sets are related by $|T' \cup T^0| = 2 \cdot |\mathcal{P}'| - 1$.*

Proof. From the description of the algorithm, we conclude that $\mathcal{P}'$ consists of pairwise disjoint rectangles, i.e., it is a packing, and that $|T'| = |\mathcal{P}'|$. Since T^0 is the set of concave corners of the staircase $\mu(T')$ whose convex corners are a subset of the points of T' we obtain that $|T^0| \leqslant |T'| - 1$. By definition $T' \cup T^0$ is a hitting set of $\mathcal{R}'$. $\square$

Now, we continue with the basic property of the family $\mathcal{R}''$.

Proposition 1. *The family $\mathcal{R}''$ is $\|$-cross separable with respect to $\mu := \mu(T')$.*

Proof. From the definition of $\mathcal{R}_b$ we conclude that each point of T' is below the axis-monotone curve γ. Thus the upper zigzag μ of T' is also below γ. Let $R'' \in \mathcal{R}''$ and let R be the rectangle of $\mathcal{P}'$ because of which R'' was inserted into $\mathcal{R}''$, i.e., $R'' \cap R \neq \emptyset$ and because R'' remains in $\mathcal{R}''$ also $R'' \cap (T' \cup T^0) = \emptyset$.

The remaining part of the proof is split into three claims.

Claim 1. *The bottom side of R is at least as high as the bottom side of R'' and the right side of R is to the right of the right side of R''.*

Proof of Claim 1. The statement about the bottom sides is due to the choice of R in Step 2 of the algorithm. The statement about the right sides then follows from $R'' \cap R \neq \emptyset$ and $c_R \notin R''$. $\square$

Claim 2. *If μ intersects a rectangle $R'' \in \mathcal{R}''$, then μ necessarily $\|$-crosses R''.*

Proof of Claim 2. Since R'' is not removed from $\mathcal{R}''$ at Step 8, R'' contains no corner of the zigzag μ. Therefore, μ either $\|$-crosses or $=$-crosses R''. Suppose by way of contradiction that R'' and μ $=$-cross. Let s be the vertical segment of μ traversing R''. Let c be the lower extremity of s, and let c'' be the bottom right corner of R''. Note that the intersection of s and R'' implies that $c \leqslant c''$ in dominance, i.e., componentwise. If R is the rectangle intersecting R'' because of which R'' was inserted into $\mathcal{R}''$, then it follows from Claim 1 that $c'' \leqslant c_R$ in dominance. By transitivity $c \leqslant c_R$ in dominance. This contradicts the fact that c is a corner of the upper zigzag $\mu(T')$ with $c_R \in T'$. $\square$

Claim 3. *μ intersects all rectangles of $\mathcal{R}''$.*

Proof of Claim 3. Suppose by way of contradiction that $R'' \cap \mu = \emptyset$ for some $R'' \in \mathcal{R}''$. If R'' is above μ, from Claim 1 we conclude that the lowest right corner $c_R \in T'$ of R is also above μ. This is in contradiction to the definition of μ as the upper zigzag of T'. Therefore, R'' is below μ but since μ is below γ we find by transitivity that R'' is below γ. This is in contradiction to the fact that γ is the curve certifying that the family $\mathcal{R}$ is separable. This contradiction establishes Claim 3 and concludes the proof of the proposition. $\square$

We can now conclude the proof of Theorem 2. A call of HittingSet($\mathcal{R}_b$) returned a partition $\mathcal{R}' \cup \mathcal{R}''$ of $\mathcal{R}_b$. By Proposition 1 the family $\mathcal{R}''$ is cross separable, hence, its hitting set T'' and packing $\mathcal{P}''$ are of equal size. From the construction we know that $T' \cup T^0$ is a hitting set and $\mathcal{P}'$ is a packing for $\mathcal{R}'$. Their sizes are related by the inequality $|T' \cup T^0| \leqslant 2|\mathcal{P}'|$ (Lemma 4).

From Lemma 1 we obtain that $T'' \cup T' \cup T^0$ is a hitting set of $\mathcal{R}_b$ of size at most $3\tau(\mathcal{R}_b)$ and that the larger of the two packings $\mathcal{P}'$ and $\mathcal{P}''$ is a packing of $\mathcal{R}_b$ of size at least $\nu(\mathcal{R}_b)/3$. Part (c) of the lemma implies the inequality $\tau(\mathcal{R}_b) \leqslant 3\nu(\mathcal{R}_b)$.

Theorem 1 follows easily. The original set $\mathcal{R}$ of rectangles was partitioned as $\mathcal{R}_b \cup \mathcal{R}_r$. With two calls of HittingSet we obtain hitting sets T_b and T_r and packings $\mathcal{P}_b$ and $\mathcal{P}_r$ for these families that differ in size by a factor of at most 3. From Lemma 1 we obtain that $T_b \cup T_r$ is a hitting set of $\mathcal{R}$ of size at most $6\tau(\mathcal{R})$. The larger of the two packings $\mathcal{P}_b$ and $\mathcal{P}_r$ is a packing of $\mathcal{R}$ of size at least $\nu(\mathcal{R})/6$. And finally $\tau(\mathcal{R}) \leqslant 6\nu(\mathcal{R})$.

It remains to show that testing if a family $\mathcal{R}$ of n axis-parallel rectangles is separable and the algorithm can be implemented is $O(n \log n)$. Below we sketch how to do this using standard techniques like plane sweep algorithms and segment trees that can be found in most text books on computational geometry, e.g. [9].

To check whether $\mathcal{R}$ is separable with an axis-monotone curve γ from north-west to south-east it is enough to scan the input with a sweep line algorithm. The sweep computes the upper zigzag $\mu(B)$ of the set B of bottom left corners and the lower zigzag $\lambda(A)$ of the set A of top right corners. The input family $\mathcal{R}$ is separable exactly if for every x-coordinate $\mu_x(B) \leqslant \lambda_x(A)$; if so we can use $\lambda(A)$ or any other monotone curve that stays between $\mu(B)$ and $\lambda(A)$ as the separating curve γ for the algorithm. The complexity of the algorithm is $O(n \log n)$.

To partition $\mathcal{R}$ into $\mathcal{R}_b$ and $\mathcal{R}_r$ we only need to know whether the bottom right corner of $R \in \mathcal{R}$ is above or below γ. This information can be available from the computation of γ or it can be produced with a new sweep.

Finally, consider the complexity of the algorithm HittingSet($\mathcal{R}$). To find the rectangle with the highest bottom side we keep a list with all rectangles sorted by decreasing bottom side. In the run of the algorithm this list is traversed once.

Lemma 5. *The overall running time for Step* 4 *can be bounded by* $O(n \log n)$.

Proof. The efficient execution of Step 4 will be based on the following observation concerning the x-projections $I'' = [x''_l, x''_r]$ of R'' and $I = [x_l, x_r]$ of R. If $R'' \cap R \neq \emptyset$, then $I'' \cap I \neq \emptyset$. Conversely if $x_l \leqslant x''_r \leqslant x_r$, then $R'' \cap R \neq \emptyset$ and if $x_l \leqslant x''_l \leqslant x_r$, then $R'' \cap R \neq \emptyset$ if and only if the top side of R'' is at least as high as the bottom side of R.

We store the x-projections of the rectangles in a segment tree. A node N of this tree corresponds to an interval (a, b), i.e., $N = N(a, b)$ and at $N(a, b)$ we store a set of intervals containing (a, b) in a list that is sorted by decreasing upper end of the corresponding rectangle. To find the rectangles intersecting R we make a query for intervals containing x_l in the segment tree. All the rectangles corresponding to the intervals containing x_l intersect R and are removed from the data structures. This is followed with a second query with x_r, this time only an initial part of the elements stored at a traversed node are removed.

It remains to remove all the rectangles R'' with $x_l \leqslant x''_l \leqslant x''_r \leqslant x_r$. If we associate the point $p_R = (-x_l, x_r) \in \mathbb{R}^2$ with rectangle R we only have to find all rectangles R'' whose associated point is dominated by p_R. This is a simple instance of an orthogonal range query.

The initialization of the data structures can be done in $O(n \log n)$ Each query takes time $O(\log n + k)$ where k is the number of rectangles found for deletion. The deletion of a rectangle from the data structures can be done with $O(\log n)$ operations. This yields an overall running time of $O(n \log n)$ for Step 4. $\square$

Computing the upper zigzag $\mu(T')$ can again be done with a sweep. This same sweep can be used to identify those rectangles that stay in $\mathcal{R}''$, these are the rectangles that $\|$-cross the zigzag $\mu(T')$. Step 8 is nothing but the computation of a minimal clique cover and a maximum independent set of an interval graph. If the endpoints of the intervals are given in sorted order this can be done with a greedy approach in linear time.

For the call of HittingSet($\mathcal{R}$) this yields a total running time of $O(n \log n)$ and the proof of Theorems 1 and 2 is complete.

4. Open questions

Our work leaves some open questions:

1. What is the complexity of computing $\tau(\mathcal{R})$ and/or $\nu(\mathcal{R})$, for a separable family $\mathcal{R}$, of rectangles? We suspect that it is NP-hard.
2. Do $\tau(\mathcal{R})$ and/or $\nu(\mathcal{R})$, for a separable family $\mathcal{R}$, admit a PTAS?
3. What is the best possible factor c such that $\tau(\mathcal{R}) \leqslant c\nu(\mathcal{R})$ for a separable family $\mathcal{R}$? So far we know $3/2 \leqslant c \leqslant 6$.

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