Geometriae Dedicata **69:** 237–240, 1998.

A Note on Circular Decomposable Metrics

VICTOR CHEPOI* and BERNARD FICHET
Laboratoire de Biomathématiques, Université d'Aix Marseille II, 27 Bd Jean Moulin, F-13385 Marseille Cedex 5, France

(Received: 27 October 1996)

Abstract. A metric d on a finite set X is called a *Kalmanson metric* if there exists a circular ordering ζ of points of X, such that $d(y,u)+d(z,v) \geqslant d(y,z)+d(u,v)$ for all crossing pairs yu and zv of ζ. We prove that any Kalmanson metric d is an l_1-metric, i.e. d can be written as a nonnegative linear combination of split metrics. The splits in the decomposition of d can be selected to form a circular system of splits in the sense of Bandelt and Dress.

Mathematics Subject Classifications (1991): Primary 52A01, 52A30.

Key words: l_1-metric, circular decomposable metric, Crofton formula.

An l_1*-metric (decomposable metric)* [6] d on a finite set X is any nonnegative linear combination of split metrics $d = \Sigma_{S \in \mathcal{S}} \alpha_S \cdot \delta_S$, where the *split (pseudo) metric* δ_S associated with the split (bipartition) $S = \{A, B\}$ of X is defined as

$$\delta_S(x,y) = \begin{cases} 0, & \text{if } x,y \in A \text{ or } x,y \in B, \\ 1, & \text{otherwise.} \end{cases}$$

A *circular ordering* $\zeta = (x_0, \dots, x_n = x_0)$ of $X, |X| = n$, is a bijection between X and the vertices of a convex n-gon C_n in the plane such that x_i and $x_{i+1} (i = 0, 1, \dots, n-1)$ are consecutive vertices of C_n. For each pair x_i, x_j of distinct points of X let $S_{ij} = \{A_{ij}, B_{ij}\}$, where $A_{ij} = \{x_{i+1}, \dots, x_j\}$ and $B_{ij} = \{x_{j+1}, \dots, x_i\}$. Then $\mathcal{S}(\zeta) = \{S_{ij} : x_i, x_j \in X\}$ will be called the *circular system* of splits associated with ζ. Now, an l_1-metric d is *circular decomposable* if there exists a circular ordering ζ of X and a decomposition of d such that every split occurring in this decomposition belongs to $\mathcal{S}(\zeta)$. Circular decomposable metrics have been introduced by Bandelt and Dress [2] in their study of totally decomposable metrics. They note that tree metrics and the Euclidean metric restricted to points that form a convex polygon in the plane are circular decomposable.

A (pseudo) metric d on X is a *Kalmanson metric* if there exists a circular ordering ζ of X, that

$$d(y,u) + d(z,v) \geqslant d(y,z) + d(u,v), \tag{1}$$

for all $u, v, y, z \in X$ such that the segments $[y,u]$ and $[z,v]$ are crossing diagonals of C_n (following [2] we write $yu|zv$ for pairs constituting crossing diagonals). In

* On leave from the Universitatea de stat din Moldova, Chişinău.

this case, ζ and d are called *compatible*. These metrics have been introduced in [7] and further investigated in [5]. Notice that all Kalmanson metrics compatible with a given circular ordering ζ of X form a polyhedral cone.

In this note we show that Kalmanson metrics and circular decomposable metrics are the same. Since this result was obtained, we learned about the paper [4] by Christopher, Farach and Trick. Using some results from [2] and [5], these authors also establish the equivalence between the two classes of metrics. However, our proof is very short, self-contained, and indicates the required circular decomposition. To show that Kalmanson metrics are circular decomposable, we prove a Crofton-type formula for computing distances in metric spaces. (Recall that the Crofton formula from integral geometry establishes the probability that a line would separate two planar convex sets.) The idea is borrowed from a striking paper by R. Alexander [1].

THEOREM. *Let d be a metric on a finite set X. Then d is a Kalmanson metric if and only if d is circular decomposable.*

Proof. We start with a few notions. As before, let ζ be a circular ordering of X. A split $S_{ij} = \{A_{ij}, B_{ij}\}$ *separates* the points $u, v \in X$ if $u \in A_{ij}$ and $v \in B_{ij}$ or $u \in B_{ij}$ and $v \in A_{ij}$. For a split S_{ij}, the pairs $x_i x_j, x_i x_{j+1}, x_{i+1} x_j$, and $x_{i+1} x_{j+1}$ will be called *extreme pairs* of S_{ij}.

First, let d be circular decomposable, i.e. $d = \Sigma \alpha_{S_{ij}} \delta_{S_{ij}}$ for a circular ordering ζ of X and $\alpha_{S_{ij}} \geqslant 0$. To establish that d is Kalmanson, it suffices to show that any split metric $\delta_{S_{ij}}$ is a Kalmanson metric compatible with ζ. Pick $y, z, u, v \in X$ with $yu|zv$. If S_{ij} separates y and u and/or z and v, then (1) can be verified in a straightforward way. In the remaining case, we deduce that all four points belong to a common part of the split S_{ij}. In this case, $d(y,u) = d(z,v) = d(y,z) = d(u,v) = 0$, and we are done. Therefore, d is a Kalmanson metric compatible with ζ.

The converse follows from a more general result. Namely, let $d\colon X \times X \to \mathbb{R}$ be a symmetric function which vanishes on the main diagonal, and let ζ be a circular ordering of X. For a split $S_{ij} \in \mathcal{S}(\zeta)$ set

$$\alpha_{ij} = d(x_i, x_j) + d(x_{i+1}, x_{j+1}) - d(x_i, x_{j+1}) - d(x_{i+1}, x_j).$$

LEMMA. *For any $u, v \in X$ there is a combinatorial Crofton formula given by*

$$2d(u,v) = \Sigma\{\alpha_{ij}\colon S_{ij} \text{ separates the points } u \text{ and } v\}. \tag{2}$$

Proof. We show that a pair $x_i x_j \neq uv$ occurs as an extreme pair in such a way that $+d(x_i, x_j)$ and $-d(x_i, x_j)$ appear the same number of times on the right side of (2), while $+d(u,v)$ appears exactly twice. Three main cases have then to be distinguished.

First suppose that x_i, x_j, u, v are distinct points. If the split S_{ij} does not separate the points u and v, then $d(x_i, x_j)$ cannot appear on the right side of (2).

Now, let $u \in A_{ij}$ and $v \in B_{ij}$. Then $x_i x_j$ occurs as extreme pair of four splits $S_{ij}, S_{i-1j-1}, S_{ij-1}, S_{i-1j}$ separating the points u and v. Then the first two splits will contribute $+d(x_i, x_j)$ while the last two splits will contribute $-d(x_i, x_j)$.

Now, suppose $x_i = u$, but $x_j \neq v$. Let $v \in A_{ij}$. There are exactly two splits S_{ij} and S_{ij-1} which separate u and v in such a way that $x_i x_j$ is an extreme pair. The first split gives a contribution $+d(x_i, x_j)$ in (2) and the second gives $-d(x_i, x_j)$.

Finally, let $x_i = u$ and $x_j = v$. There are precisely two splits S_{ij} and S_{i-1j-1} which separate u and v, and $x_i x_j$ occurs as an extreme pair of each of them. Each of these splits has a contribution $+d(x_i, x_j) = +d(u, v)$ in (2). This concludes the proof of the lemma.

If d is a Kalmanson metric then every α_{ij} is nonnegative, i.e. (2) yields a circular decomposition of d. □

Clearly, the set of functions d occurring in the lemma forms a $n(n-1)/2$-dimensional real vector space. In this space, formula (2) shows that any d may be written as $d = \frac{1}{2}\Sigma\alpha_{ij} \cdot \delta_{S_{ij}}$. Therefore, the split metrics $\delta_{S_{ij}}$ span the space. Since their number is $n(n-1)/2$, they form a basis and the decomposition given by (2) is unique. Further, the cone of Kalmanson metrics compatible with a given circular order is the nonnegative orthant of the corresponding basis. The whole set of Kalmanson metrics is the union of all such cones taken over all circular orderings. Note that this union is nonconvex for $n \geqslant 5$ and that the cones share a common face spanned by the split metrics $\delta_{\{\{x\}, X-\{x\}\}}$.

A metric d on X is called *convex* if for any distinct points $x, y \in X$ there exists a point $z \neq x, y$ such that $d(x, y) = d(x, z) + d(z, y)$. It is well known [8] that if, in addition, (X, d) is complete, then any two points $x, y \in X$ can be joined by a shortest path $P(x, y)$ which is a Jordan curve.

A metric space (X, d) is said to be L_1*-embeddable* if there is a measurable space $(\Omega, \mathcal{A})$, a nonnegative measure μ on it and a mapping λ of X into the set of measurable functions F (i.e. with $\|f\|_1 = \int_\Omega |f(w)|\mu(dw) < \infty$) such that

$$d(x, y) = \|\lambda(x) - \lambda(y)\|_1$$

for all $x, y \in X$ [6]. A well-known compactness result of [3] implies that L_1-embeddability of a metric space is equivalent to l_1-embeddability of its finite subspaces.

COROLLARY. *Let $\mathcal{D}$ be a topological two-disc bounded by a Jordan curve Γ. If d is a convex metric on $\mathcal{D}$, then the metric space (Γ, d) is L_1-embeddable.*

Proof. By the result of [3] and the theorem, it suffices to establish that d restricted to any finite subset X of Γ is a Kalmanson metric. Let ζ be a circular ordering of X obtained by walking around the curve Γ. Pick points $y, z, u, v \in X$ such that $yu|zv$ in ζ. Since $(\mathcal{D}, d)$ is a complete metric space, the pairs y, u and z, v can be joined in $\mathcal{D}$ by shortest paths $P(y, u)$ and $P(z, v)$ which are Jordan curves. Necessarily, $P(y, u)$ and $P(z, v)$ intersect in a point p. By the triangle inequality

$d(y,z) \leqslant d(y,p)+d(z,p)$ and $d(u,v) \leqslant d(u,p)+d(v,p)$. Summing up we obtain (1), i.e. d is Kalmanson. □

We conclude with another example of a Kalmanson metric d on $X = \{x_1, x_2, \ldots, x_n\}$: let $a_1 \leqslant a_2 \cdots \leqslant a_n$ and define $d(x_i,x_j) = |a_j - a_i| \cdot (a_n - |a_j - a_i|)$.

Acknowledgement

We would like to thank Monique Laurent for acquainting us with the paper of Christopher, Farach, and Trick.

References

1. Alexander, R.: Planes for which the lines are the shortest paths between points, *Illinois J. Math.* **22** (1978), 177–190.
2. Bandelt, H.-J. and Dress, A. W. M.: A canonical decomposition theory for metrics on a finite set, *Adv. Math.* **92** (1992), 47–105.
3. Bretagnolle, J., Dacunha Castelle, D., and Krivine, J.-L.: Lois stables et espaces L^p, *Ann. Institute Henri Poincaré sect. B* **2**(3) (1966), 231–259.
4. Christopher, G., Farach, M., and Trick, M. A.: The structure of circular decomposable metrics, In: J. Diaz and M. Serna (eds), *Algorithms-ESA'96*, Lecture Notes in Computer Science 1136 Springer, New York, 1996, pp. 486–500.
5. Deineko, V., Rudolf, R. and Woeginger, G.: Sometimes traveling is easy: the master tour problem, Technical Report, Institute of Mathematics, University of Technology, Graz (1995).
6. Deza, M. and Laurent, M.: *Geometry of Cuts and Metrics*, Springer, Berlin, 1997.
7. Kalmanson, K., Edgeconvex circuits and the travelling salesman problem, *Canadian J. Math.* **27** (1975), 1000–1010.
8. Rinow, W.: *Die innere Geometrie der metrischen Räume*, Springer, Berlin, 1961.