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The algebra of metric betweenness I: Subdirect representation and retraction

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Abstract

We bring together algebraic concepts such as equational class and various concepts from graph theory for developing a structure theory for graphs that promotes a deeper analysis of their metric properties. The basic operators are Cartesian multiplication and gated amalgamation or, alternatively, retraction. Specifically, finite weakly median graphs are known to be decomposable (relative to these operators) into smaller pieces that in turn are parts of hyperoctahedra, the pentagonal pyramid, or of certain triangulations of the plane. This decomposition scheme can be interpreted as Birkhoff's subdirect representation in purely algebraic terms.

0. Introduction

Structure theories for graphs that directly allude to algebraic concepts, such as variety and subdirect representation, have been developed rather sporadically. In universal algebra, a variety (alias primitive class) is a class of algebras endowed with any finitary operations that is closed under taking homomorphic images, subalgebras and (direct) products. By Birkhoff's theorem, varieties are exactly the equational classes, i.e. they consist of all algebras of the same type satisfying a (possibly infinite) number of equations [38,57]. In graph theory, varieties have rather been understood to be classes closed under retracts and products, but then an equational theory is not necessarily in sight. The choice of product is not unique here: it could e.g. either be the strong product or the Cartesian product [42]. In the former case, absolute retracts (of reflexive graphs)

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come into play [40], whereas for the latter product one is dealing with median graphs and their generalizations [12]. Subdirect representation in the algebraic context is an embedding into a product such that the projections onto the factors are surjective. By another theorem of Birkhoff, every algebra admits a subdirect representation by subdirectly irreducible algebras (characterized as the algebras for which the nontrivial congruences do not intersect in the equality relation). There have been attempts to imitate subdirect representation for categories of graphs ([39,47]), but they seem to be somewhat too general (i.e. with too few subdirectly irreducibles) in order to be useful for specific graph-theoretic questions. There were also approaches to go from graphs to algebras: the so-called graph algebras, introduced in [48], use a (quasi-trivial) partial binary operation on the vertex set of a graph for codifying the edges that is extended to a full operation by adjoining a zero. This has served as a framework for constructing algebras with unusual properties (cf. [29]) rather than for a deeper understanding of graphs.

Some classes of graphs, possessing distinctive features of the geometry of their shortest paths, can be interpreted in algebraic terms quite naturally. Median graphs and their algebras constitute the simplest instance: the ternary operation on the vertex set associates to each triplet u, v, w the unique *median* $x = (uvw)$, i.e. the vertex x lying simultaneously on shortest paths between the three pairs from the triplet; see [1,2,49–51], cf. [9]. The median algebras resulting from this association are all subdirect products of the two-element algebra K_2. Extending the list of subdirectly irreducible algebras in order to encompass all complete graphs then yields quasi-median algebras or isotropic media [12,43]. What is remarkable in this context is that the purely algebraic avenue leads to objects that can be regarded as particular graphs in the finite case, which admit quite a number of alternative characterizations. Now that quasi-median graphs have been generalized further to weakly median graphs, thereby maintaining a decomposition into simple building stones (“prime graphs”) [7], one may wonder whether the algebra goes along with it, too. Somewhat surprisingly, it does—although the list of prime graphs includes quite different types of graphs: induced subgraphs of hyperoctahedra and triangulations of certain plane graphs. It is perhaps no accident that the prime models all have some geometric interpretation.

The paper is organized as follows. In the next section, definitions are provided that are necessary for dealing with weakly median graphs and their prime constituents. Then Section 2 describes the intrinsic algebras associated with any graph. Particular interest attaches to those (“apiculate”) graphs for which there is a unique intrinsic algebra. Some basic equations can readily be established for the corresponding ternary algebras. In Section 3 we prove the main result of this paper (Theorem 1 together with Corollary 2): successive gated amalgamations lead to the subdirect representation of the resulting associated algebra by subdirectly irreducibles whenever they begin with a class of (“prime”) graphs that possess only trivial gated subgraphs (and hence yield simple algebras). If these prime building stones have some additional properties, all fulfilled for prime weakly median graphs, then this subdirect representation can also be interpreted in terms of retracts (and Cartesian products), as is demonstrated in Section 4 (Corollary 4). Although for infinite weakly median graphs one does not necessarily have a finite decomposition scheme, the subdirectly irreducibles (viz., the prime constituents) can be retrieved as the weak Cartesian factors of the blocks relative to a canonical tolerance (Theorem 2 in Section 5). Much of this follows from the elegant theory of fiber-complementedness developed by Chastand [19].

In the follow-up paper (part II), we further elaborate on the geometric structure of weakly median graphs, especially in the planar case. This enables us to establish equations in four variables in terms of the ternary imprint operation, by which we can eventually characterize weakly median graphs in a purely equational way among discrete ternary algebras.

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1. Preliminaries

All graphs $G = (V, E)$ occurring here are undirected, connected, and without loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any of its vertices. By the *convex hull* $\mathrm{conv}(W)$ of W in G we mean the smallest convex subset of V (or induced subgraph of G) that contains W. An *isometric subgraph* of G is an induced subgraph in which the distances between any two vertices are the same as in G. In particular, convex subgraphs are isometric. A graph G is *weakly modular* [6,11,22] if its distance function d satisfies the following conditions:

Triangle condition (T): for any three vertices u, v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

Quadrangle condition (Q): for any four vertices u, v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

These two conditions are fulfilled by modular [14], pseudo-modular [10], quasi-median graphs [12,42], pre-median graphs [19], incidence graphs of dual polar spaces [17], and bridged graphs [33,52]. Recall that a graph is called *bridged* if it does not contain any isometric cycle of length greater than 3, or alternatively, if the *neighborhood* $N(A) = \{y \in V : y$ is adjacent to some $x \in A\}$ of every convex set A of G is convex. (T) and (Q) can be merged into a single condition; namely, a graph G is weakly modular if and only if it satisfies

(TQ): for any three vertices u, v, w such that v and w are at distance 2 and have some common neighbor z with $2d(u, z) > d(u, v) + d(u, w)$, there exists a common neighbor x of v and w with $2d(u, x) < d(u, v) + d(u, w)$.

Indeed, (Q) is a trivial consequence of (TQ), and in order to derive (T) from (TQ) proceed by induction on $d(u, v) = d(u, w)$. First choose some neighbor v' of v in $I(u, v)$: if v' is also adjacent to w, then v' is a vertex as required by (T). Otherwise, apply (TQ) to the triplet u, v', w, which then yields a common neighbor w' of v' and w such that $d(u, v') = d(u, w')$. Next apply (TQ) to the triplet v, w, u' where u' is the common neighbor of v' and w' in $I(u, v')$. This establishes (T). Conversely, in a weakly modular graph, (TQ) evidently holds for those triplets u, v, w where $d(u, v) = d(u, w)$, by virtue of (Q). Now, if $d(u, w) = d(u, v) + 1$ and $2d(u, z) > d(u, v) + d(u, w)$ for some common neighbor z of v and w, then employing (T) and (Q) yields vertices y and t such that t is a neighbor of v in $I(u, v)$ and y is a common neighbor of w, z, and t; the required vertex x is a common neighbor of t, v, and w provided by (T).

A *weakly median* graph is a weakly modular graph that does not contain any two distinct vertices x, y with an unconnected triplet u, v, w of common neighbors; see Fig. 1. Weakly median graphs thus satisfy the stronger variants, $(T!)$, $(Q!)$, and $(TQ!)$, of the above conditions (T),(Q), and (TQ) which additionally require uniqueness of that neighbor x; see Fig. 2 for two (minimal) instances fulfilling (Q) but not $(Q!)$. Indeed, if a weakly modular graph violates $(T!)$ or $(Q!)$, we obtain one of the graphs of Fig. 1 as an induced subgraph; if in some instance of

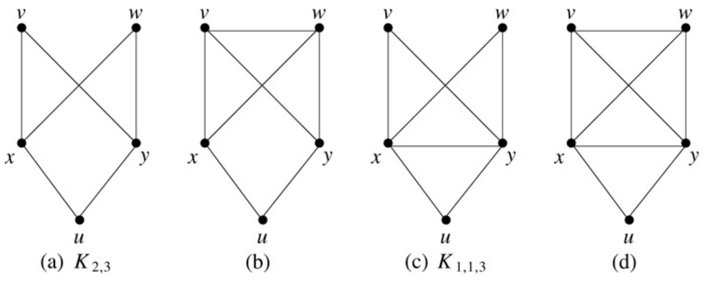


Fig. 1. Weakly modular graphs that are not weakly median.

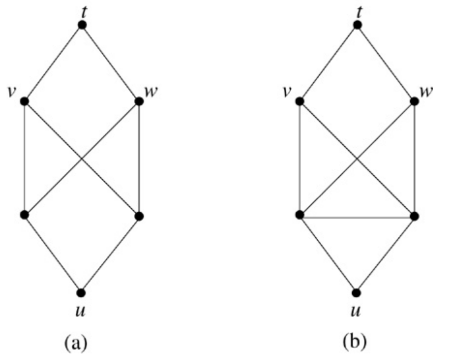
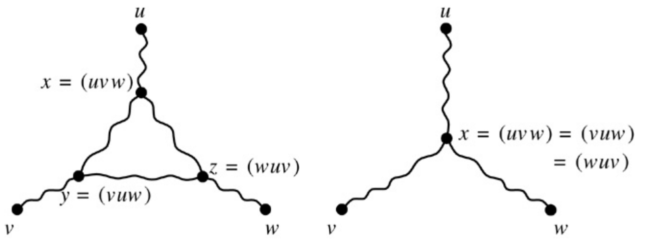


Fig. 2. Graphs fulfilling (Q) but not $(Q!)$.

(TQ) there was yet another vertex x' having the same distances to u, v, w as x, then either $(Q!)$ would be violated or v, w, together with some common neighbor t of x and x' in $I(u, x)$ would constitute an unconnected triplet of common neighbors for x and x'. Since, on the other hand, $(TQ!)$ rejects any of the four graphs indicated in Fig. 1, we can therefore state that a graph G is weakly median if and only if it satisfies $(TQ!)$.

An induced subgraph H of a graph G is called *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'; cf. [31]. G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. In regard to a decomposition scheme involving multiplication and amalgamation, a graph with at least two vertices is said to be *prime* if it is neither a proper weak Cartesian product [42] nor a gated amalgam of smaller graphs. For instance, the only prime median graph is the two-vertex complete graph K_2; see [43,55]. More generally, the prime quasi-median graphs are exactly the complete graphs [12,43]; see [42] for more information about quasi-median graphs. In [7], we established that the prime weakly median graphs are precisely (i) the *5-wheel* (a 5-cycle plus a pivot vertex adjacent to all vertices of the cycle), (ii) the *subhyperoctahedra* (induced subgraphs of hyperoctahedra, that is, multipartite graphs of the form $K_{i_1,i_2,i_3,\ldots}$ with $1 \le i_j \le 2$) different from the singleton graph K_1, the 3-vertex path $P_2 = K_{1,2}$, and the 4-cycle $C_4 = K_{2,2}$, and (iii) the *2-connected K_4- and $K_{1,1,3}$-free bridged graphs*, which are exactly the graphs embeddable in the plane such that all inner faces are triangles and all inner vertices have degrees larger than 5.



(a) Non-symmetric. (b) Symmetric.

Fig. 3. Apex operation.

2. Intrinsic algebras and apiculate graphs

Every graph G with vertex set V can be turned into a ternary algebra, called an *apex algebra* of G [12]: an *apex operation* $(\ldots) : V^3 \to V$ maps any triplets u, v, w and u, w, v to a vertex $x = (uvw) = (uwv) \in I(u, v) \cap I(u, w)$, called a u-apex relative to v and w, such that $I(u, x)$ is maximal with respect to inclusion; see Fig. 3 for an illustration (where the twiggled lines and their concatenations via intermediate vertices indicate shortest paths). Consequently,

$$I(u, v) = \{(uvx) : x \in V\} = \{(uxv) : x \in V\},$$
$$I((uvw), v) \cap I((uvw), w) = \{(uvw)\}.$$

The following equations then trivially hold:

(A1) $(uvv) = v$ (right majority),
(A1′) $(uuv) = u$ (left majority),
(A2) $(uvw) = (uwv)$ (right symmetry),
(A3) $(vu(uvw)) = (uvw)$ (twisted left absorption),
(A3′) $(uv(uvw)) = (uvw)$ (left absorption),
(A4) $((uvw)vw) = (uvw)$ (right absorption),
(A4′) $((uvw)uv) = (uvw)$ (left-right absorption).

Here and in all subsequent equations for ternary operations the variables are understood to run through the whole (vertex) set, unless specified otherwise. Note that (A1′),(A2), and (A4′) are exactly the axioms of 2, 1, and 5, respectively, of Isbell [43, p. 322], whereas his axiom 4b is a consequence of (A2) and (A3). Note that (A3′) follows from (A3) because

$$(uv(uvw)) = (uv(vu(uvw))) = (vu(uvw)) = (uvw).$$

Further, (A1), (A2), and (A3) together imply (A1′):

$$(uuv) = (uvu) = (uv(vuu)) = (vuu) = u.$$

The inherent non-uniqueness of apices does not permit a canonical choice for (uvw), but at least one could employ a priority rule. Namely, assume that the vertices are enumerated by some ordinal, providing the vertex set with a priority order. Then, if the vertices u, v, w admit a median, define $(uvw) = (uwv) = (vuw) = (vwu) = (wuv) = (wvu)$ as the median of u, v, w having highest priority; else, let $(uvw) = (uwv)$, $(vuw) = (vwu)$, and $(wuv) = (wvu)$ each be a respective apex of highest priority. We will refer to the resulting apex algebra as to a *priority* apex algebra.

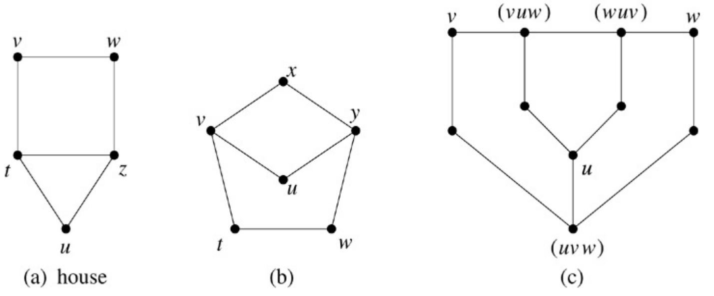


Fig. 4. Apiculate graphs.

When a graph G admits several distinct apex operations, these operations can be iterated to generate further ternary operations in the following way. Let $\nabla_u(v, w)$ be the smallest set S minus $\{v, w\}$ such that S includes v, w, and (uxy) for all $x, y \in S$ and apex operations $(\ldots)$ of G. Then any ternary operation $(\ldots)$ on V such that $(uvw) \in \nabla_u(v, w)$ is called an *intrinsic operation* of G, and the set V together with this operation is an *intrinsic algebra* of G. The interval $I(u, v)$ can be recovered from any intrinsic algebra just as in the case of an apex algebra. Observe that any intrinsic operation satisfies the above equations (A1)–(A4′) except possibly (A4). Trivially, there exists a unique vertex $s \in \nabla_u(v, w)$, referred to as the *imprint* of v, w with respect to u, that is at minimal distance to u, satisfying $s \in I(u, t)$ for all $t \in \nabla_u(v, w)$. For example, the imprint of v, w with respect to u in the graphs of Fig. 1 equals u, whereas an apex operation would select either x or y for this particular triplet u, v, w. The *imprint operation* of G then assigns to each triplet u, v, w the imprint of v, w with respect to u. It fulfills all of the above equations including (A4). This imprint function was introduced by Feder [34,35] as the appropriate generalization of the imprint function of a quasi-median graph [26,42] to arbitrary graphs. A different generalization is used in [16] under the same name, which constitutes the "median function" m of Tardif [54]: $m(u, v, w)$ is the gate of u in the smallest gated set containing v and w; see Lemma 1(e) below. Particular interest attaches to the case where imprint and apex operations coincide, that is, when the graph possesses a unique intrinsic algebra. In this case, we say that the graph is *apiculate*; see Fig. 4 for examples. In other words, G is apiculate if and only if for any vertex a the vertex set of G is a meet-semilattice with respect to the base-point order $\preceq_a$ defined by $u \preceq_a v \Leftrightarrow u \in I(a, v)$, that is, $I(a, v) \cap I(a, w) = I(a, (avw))$. Then every principal ideal $I(a, b)$ of the meet-semilattice $(V, \preceq_a)$ is a lattice, where the b-apex relative to $w, x \in I(a, b)$ is their join. Each of these lattices is modular when G is weakly median. Indeed, the quadrangle and triangle conditions imply that $(I(a, b), \preceq_a)$ is lower and upper semimodular, which is equivalent to modularity because of finite length; see [27]. This (semi)lattice condition alone, of course, does not characterize weakly median graphs. For instance, all base-point orders in a geodesic graph (e.g. an odd cycle or the Petersen graph) yield tree semilattices, whence geodesic graphs are apiculate.

Proposition 1. *A graph G is apiculate if and only if some intrinsic operation of G satisfies one of the equations*

(A5) $(uv(uwx)) = (u(uvw)x)$ (*associativity*),
(A5′) $(u(uvw)(uv(uwx))) = (uv(uwx))$ (*monotonicity*).

Proof. In order to verify (A5) for an apiculate graph G, set $a := (uv(uwx))$ and $b := (u(uvw)x)$. Note that $a, b \in I(u, v) \cap I(u, w) \cap I(u, x)$. Since (uvw) and (uwx) are the u-apices relative to

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v, w and w, x, respectively, we also obtain that

$$a, b \in I(u, (uvw)) \cap I(u, (uwx)).$$

Hence $b \in I(u, v) \cap I(u, (uwx))$. Since a is the u-apex relative to v and (uwx) and the graph G is apiculate, we conclude that $b \in I(u, a)$. Analogously, one can show that $a \in I(u, b)$. Evidently, this implies that $a = b$.

When (A5) is satisfied, then in the particular instance

$$(u(uvw)(u(uvw)x)) = (u(uvw)x)$$

of (A3′) we can replace $(u(uvw)x)$ by $(uv(uwx))$ and thus obtain (A5′). Hence (A5) implies (A5′).

Finally, if (A5′) holds, then consider any u-apex x relative to v and w. Then $(u(uvw)x) = x$ holds by (A5′), whence $x = (uvw)$ follows and therefore G is apiculate. □

A graph G is a *Pasch graph* [8,56] if it satisfies the following analogue of the Pasch axiom of elementary geometry: for any five vertices u, v, w, x, y with $x \in I(u, v)$ and $y \in I(u, w)$, the intervals $I(v, y)$ and $I(w, x)$ intersect. This in turn is equivalent to the requirement that for any three vertices u, v, w the *(interval-)shadow*

$$I(v, w)/u = \{x \in V : I(u, x) \cap I(v, w) \neq \emptyset\}$$

is convex [23]. Recall that a subset A of V is called *convex* if $I(x, y) \subseteq A$ for all $x, y \in A$. The key feature of Pasch graphs is the following separation property (by which they are actually characterized): each pair of disjoint convex sets can be extended to a pair of complementary convex sets, called *halfspaces*; see [23]. Since weakly median graphs are exactly the weakly modular Pasch graphs [23], all subsequent properties established for Pasch graphs thus hold for weakly median graphs as well. For instance, intervals $I(u, v)$ are convex [56, Proposition 4.15] and, trivially, every *(point-)shadow* $v/u = \{v\}/u$ (also called extension of v from u [46]) is convex in a Pasch graph.

Proposition 2. *Every Pasch graph G is apiculate.*

Proof. Pick an arbitrary triplet u, v, w and let x be a vertex in $I(u, v) \cap I(u, w)$ furthest away from u. Suppose by way of contradiction that there exists a vertex $y \in I(u, v) \cap I(u, w)$ outside the interval $I(u, x)$. By the Pasch axiom, the intervals $I(x, w)$ and $I(y, v)$ have a vertex z in common. From the choice of x and y we infer that $z \neq x, y$. Since $x, y \in I(u, v) \cap I(u, w)$ and $z \in I(x, w) \cap I(y, v)$, we conclude that $x, y \in I(z, u)$ and $z \in I(u, v) \cap I(u, w)$, contrary to the choice of x and y. □

By this observation, the weakly modular apiculate graphs are exactly the weakly median graphs (as the four forbidden five-vertex graphs are not apiculate). The Petersen graph shows that an apiculate graph is not necessarily a Pasch graph even when intervals and point-shadows are convex. The latter condition can be turned into an equation, as we see next.

Proposition 3. *An apiculate graph has convex point-shadows exactly when its imprint operation satisfies*

(A6) $(u(uvw)(vwx)) = (uvw)$.

Proof. Assume that (A6) holds. If $v, w \in y/u$ and $x \in I(v, w)$, then

$$(u(uvw)x) = (u(uvw)(vwx)) = (uvw),$$

whence $x \in (uvw)/u \subseteq y/u$ because G is apiculate. Conversely, convexity of $(uvw)/u$ yields $(vwx) \in (uvw)/u$, which is expressed by (A6). □

Three (not necessarily distinct) vertices x, y, z of a graph G are said to form a *metric triangle* xyz if the intervals $I(x, y)$, $I(y, z)$, and $I(z, x)$ pairwise intersect only in the common end vertices. If $d(x, y) = d(y, z) = d(z, x) = k$, then this metric triangle is called *equilateral* of *size* k. A (degenerate) equilateral metric triangle of size 0 is simply a single vertex. We say that a metric triangle xyz is a *quasi-median* of the triplet u, v, w if

$$\begin{aligned} d(u, v) &= d(u, x) + d(x, y) + d(y, v),\\ d(v, w) &= d(v, y) + d(y, z) + d(z, w),\\ d(w, u) &= d(w, z) + d(z, x) + d(x, u). \end{aligned}$$

Note that this definition is more general than the specific notion used in the context of quasi-median graphs [12,42,44] in that here quasi-medians are not necessarily equilateral (or of minimum size). Observe that, for every triplet u, v, w, a quasi-median xyz can be constructed in the following way: first select any vertex x from $I(u, v) \cap I(u, w)$ at maximal distance to u, then select a vertex y from $I(v, x) \cap I(v, w)$ at maximal distance to v, and finally select any vertex z from $I(w, x) \cap I(w, y)$ at maximal distance to w. In the case that the quasi-median is degenerate ($x = y = z$), it is a median of the triplet u, v, w.

Proposition 4. *An apiculate graph G has unique quasi-medians, that is,* (uvw), (vuw), (wuv) *form the quasi-median for any triplet* u, v, w *of vertices, if and only if*

(A7) $(u(uvw)(vuw)) = (uvw)$,

or equivalently,

(A7′) $(u(vuw)w) = (uvw)$.

Proof. First observe that (A7) and (A7′) are equivalent:

$$\begin{aligned} (u(uvw)(vuw)) &= (u(vuw)(uvw)) \quad \text{by (A2)}\\ &= (u(u(vuw)v)w) \quad \text{by (A5)}\\ &= (u(uv(vuw))w) \quad \text{by (A2)}\\ &= (u(vuw)w) \quad \text{by (A3)}. \end{aligned}$$

As noticed above, one can always construct a quasi-median of u, v, w of the form $(uvw)yz$. Hence, if u, v, w admit a unique quasi-median, then it must be of the form $(uvw)(vuw)(wuv)$. Conversely, if the latter is a quasi-median of u, v, w, then any quasi-median xyz satisfies $x \in I(u, (uvw))$ etc., so that $x = (uvw)$ etc. follows because xyz is a metric triangle. □

Observe that (A6) together with (A2) implies (A7) (simply let $x = u$), that is, an apiculate graph with convex point-shadows has unique quasi-medians. An apiculate graph with unique quasi-medians need not have convex shadows v/t (Fig. 4(b)), and an apiculate graph may have multiple quasi-medians (Fig. 4(c)).

All the properties discussed in this section (apiculate, Pasch, convexity of point-shadows, uniqueness of quasi-medians, etc.) are preserved under Cartesian multiplication (understood as a finitary operation for graphs) and gated amalgamation (for the latter, the proofs are similar to the one for the Pasch property; see [56, Theorem 5.14]).

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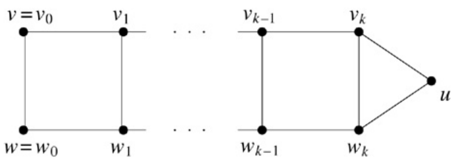

Fig. 5. k-house.

3. Gated amalgams as subdirect products

For the algebraic framework we will assume throughout (unless stated otherwise) that the graphs under consideration are endowed with their imprint operations. The vertex set V together with the imprint operation $u, v, w \mapsto (uvw)$ then constitutes the *imprint algebra* of the graph $G = (V, E)$. If the graph G has a name or an acronym, its imprint algebra will be referred to by the same name or acronym; thus, the imprint algebra of the triangle K_3 is then the triangle algebra or K_3 algebra, for short. The algebraic terms "subalgebra", "direct product", "homomorphism", "congruence", "subdirect product" etc. that refer to the imprint algebra(s) have the usual meaning [38], and we will briefly speak, for example, of the congruences etc. of the graph G that carries the imprint algebra. The direct product of graphs in this algebraic sense is, of course, traditionally referred to as the Cartesian product. If ψ is a homomorphism from G into another graph, then the congruence on G associated with ψ, called the *kernel* of ψ, is denoted by $\ker\psi = \{(x, y) \in V^2 : \psi x = \psi y\}$. A *tolerance* of G is a reflexive and symmetric binary relation ξ on V compatible with the ternary operation:

$$u\xi x,\ v\xi y,\ w\xi z \text{ implies } (uvw)\xi(xyz).$$

A *block* of ξ is any maximal set of pairwise tolerant vertices. The transitive tolerances are then the congruences. By virtue of transitivity and (A2), an equivalence relation θ is a congruence exactly when

$$v\theta w \text{ implies } (vxy)\theta(wxy) \text{ and } (xyv)\theta(xyw).$$

The congruence block containing x is denoted by $[x]$ (usually with a suffix referring to the congruence).

To give an example, consider the k-house ($k \geq 1$) in Fig. 5, generalizing the house. For any tolerance ξ of this graph different from the "all" relation ι, pairs x, y of distinct vertices can be tolerant only if $\{x, y\} \subseteq \{v_0, \ldots, v_k\}$ or $\{x, y\} \subseteq \{w_0, \ldots, w_k\}$. Indeed, $v_i \xi w_i$ for some i implies $v = (vwv_i)\xi(vww_i) = w$, whence $u = (uvw)\xi(uww) = w$, and analogously, $u\xi v$; moreover, $v = (w_k vv)\xi(w_k uv) = w_k$ and, similarly, $w\xi v_k$, yielding $\xi = \iota$ (because tolerance blocks are convex; see Lemma 1(b) below), a contradiction. In the same way, if u is tolerant with v_k or w_k, then v_k and w_k are tolerant, and we are back in the previous case. This proves the claim. It is easy to see that v_i and v_j for some $0 \leq i < j \leq k$ are tolerant exactly when w_i and w_j are tolerant. Therefore the tolerances of the k-house different from ι are in one-to-one correspondence with the tolerances of the path P_k with k edges. Hence the total number of tolerances of the k-house equals the Catalan number $\binom{2k+2}{k+1}/(k+2)$ plus 1; see [4].

The k-house is directly indecomposable (with respect to Cartesian multiplication) but it can be decomposed subdirectly. For each $1 \leq i \leq k$, the k-house G, labelled as in Fig. 5, admits a congruence θ_i with blocks $\{v_0, \ldots, v_{i-1}\}$, $\{v_i, \ldots, v_k\}$, $\{w_0, \ldots, w_{i-1}\}$, $\{w_i, \ldots, w_k\}$, and $\{u\}$. Thus, each homomorphic image G/θ_i constitutes a house. Since $\theta_1, \ldots, \theta_k$ intersect in the equality relation ω, the imprint algebra of G is embedded as a subalgebra in the product of the house algebras $G/\theta_1, \ldots, G/\theta_k$, that is, the k-house is a subdirect product of k houses (by virtue of Birkhoff's theorem; cf. [38]). The house itself is subdirectly irreducible but not simple, as it has a single nontrivial congruence.

Gated sets are readily described in terms of the imprint algebra. We say that a subset A of the vertex set V is an *ideal* if $(VAA) = \{(vab) : v \in V, a, b \in A\} \subseteq A$; note that $A \subseteq (VAA)$ holds trivially by right majority. The smallest ideal $\ll W \gg$ that contains a nonempty subset W of V is generated as follows:

$$W_0 := W \text{ and } W_k := (VW_{k-1}W_{k-1}) \text{ for } k \geq 1, \text{ so that } \ll W \gg := \cup_{k\geq 1} W_k.$$

We employ the following notation in the case of a gated amalgam $G = (V, E)$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ along their (nonempty) intersection $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. For a gated subgraph $H = (W, F)$ of G, we say that W is a *gated set* and the mapping from V to W which assigns to every vertex of G its gate in H is the *gate map* of H (and W). Since $G_1 \cap G_2$ is a gated subgraph of G, both G_1 and G_2 are gated subgraphs of G. The gates of a vertex x of G in G_1, G_2, and $G_1 \cap G_2$ are denoted by x_1, x_2, and $x' = (x_1)_2 = (x_2)_1$, respectively. For $u, v, w \in V$ one has $|\{u, v, w\} \cap V_i| \geq 2$ for some $i = 1, 2$, and therefore $(uvw) = (u_i v_i w_i)$.

Lemma 1. *Let G be a graph.*

(a) *A nonempty subset A of V is gated if and only if it is an ideal of the imprint algebra of G.*
(b) *Every block of a tolerance ξ of G is gated.*
(c) *The gate map ψ_A of a gated set A is characterized as the mapping $\psi : V \to A$ that satisfies any one of the two identities $(uvx) = (uv\psi x)$ and $(xuv) = (\psi xuv)$ for all $u, v \in A$ and $x \in V$.*
(d) *Every tolerance ξ of G is compatible with every gate map ψ_A, that is, $v\xi w$ implies $\psi_A v \xi \psi_A w$ for all $v, w \in V$.*
(e) *Every tolerance ξ of G is compatible with the ternary operation m defined by letting $m(u, v, w)$ be the gate of u in the smallest gated set $\ll v, w \gg$ containing v and w.*

Proof. (a) If A is gated, then any u-apex x relative to $v, w \in A$ necessarily coincides with its gate in A, whence $(uvw) \in \nabla_u(v, w) \subseteq A$, which proves that A is an ideal. Conversely, if A is an ideal, then for $u \in V$ choose $v \in A$ nearest to u. For any $w \in A$ we get $(uvw) \in A \cap I(u, v)$, whence $v = (uvw)$ is the gate of u in A.

(b) If $u, v, w \in V$ and $v\xi w$, then $(uvw)\xi(uvv) = v$. Moreover, for $x \in V$ with $v\xi x$ and $w\xi x$ we obtain $(uvw)\xi(uxx) = x$. This shows that every block of ξ is an ideal and hence gated by (a).

(c) Since $\nabla_u(v, x) \subseteq I(u, v) \subseteq A$ for $u, v \in A$ and $x \in V$, one obtains $\nabla_u(v, \psi_A x) = \nabla_u(v, x)$ and hence $(uvx) = (uv\psi_A x)$. Similarly, as $\nabla_x(u, v) \subseteq A$, we get $\psi_A x \in I(x, (xuv))$ and therefore $(xuv) = (\psi_A xuv)$. Conversely, if $\psi : A \to V$ satisfies at least one of the two identities, then substituting $\psi_A x$ for u and ψx for v yields $\psi_A x = \psi x$ for $x \in V$ in either case because $\psi_A x \in I(x, \psi x)$.

(d) If $v\xi w$, then indeed

$$\psi_A v = (v\psi_A v\psi_A w)\xi(w\psi_A v\psi_A w) = \psi_A w.$$

(e) By (a) and in view of the algebraic generation of $\ll v, w \gg$, the gate $m(u, v, w)$ of u can be generated from a finite number of vertices in finitely many steps by employing imprints, so that there is a polynomial function (called algebraic function in [38]) $p_{u,v,w}$ such that

$$p_{u,v,w}(r,s,t) \in \ll s,t \gg \quad \text{for all } r,s,t \in V \text{ and } p_{u,v,w}(u,v,w) = m(u,v,w).$$

Given six vertices u, v, w, x, y, z with $u\xi x$, $v\xi y$, and $w\xi z$, define another polynomial function q by

$$q(r,s,t) := (up_{u,v,w}(r,s,t)\,p_{x,y,z}(r,s,t)).$$

Then, as tolerances are compatible with all polynomial functions, we infer that $m(u,v,w)\xi m(x,y,z)$ because

$$\begin{aligned} q(u,v,w) &= (up_{u,v,w}(u,v,w)\,p_{x,y,z}(u,v,w)) = (um(u,v,w)\,p_{x,y,z}(u,v,w)) \\ &= m(u,v,w) \end{aligned}$$

and analogously $q(x,y,z) = m(x,y,z)$. □

The identities in (c) do not entail that a gate map ψ is necessarily a homomorphism. Consider, for instance, the gate map ψ from a 6-cycle to any of its edges. In fact, all edges of a bipartite graph G constitute gated subgraphs (isomorphic to K_2), but the corresponding gate maps are all homomorphisms exactly when G is a median graph.

The operation $m : V^3 \to V$ defined in (e) is in fact a *local polynomial function* [30,41] of the imprint algebra, that is, it can be interpolated by polynomial functions on all finite subsets of V^3. To see this, first note that a finitary meet operation in any meet-semilattice $(V, \preceq_a)$ is a polynomial function. Hence, with the notation in the proof of (e), we can define a polynomial function q_W for every finite subset W of V by letting $q_W(r,s,t)$ be the meet of $p_{u,v,w}(r,s,t)$ for all choices of $u, v, w \in W$ in the meet-semilattice $(V, \preceq_r)$. Then

$$q_W(x,y,z) \preceq_x p_{x,y,z}(x,y,z) = m(x,y,z)$$

and $q_W(x,y,z) \in \ll y,z \gg$ yields $q_W(x,y,z) = m(x,y,z)$ for all $x, y, z \in W$.

Lemma 2. *Let the graph G be the gated amalgam of graphs G_1 and G_2 along $G_1 \cap G_2$ such that either gate map from G_i $(i = 1,2)$ to $G_1 \cap G_2$ is a homomorphism. Then the gate map ψ of a gated set A of G is a homomorphism if and only if either gate map $\psi_i : V_i \to A \cap V_i$ for $A \cap V_i \neq \emptyset$ is a homomorphism.*

Proof. Necessity is trivial, as ψ_i is the concatenation of ψ and the gate map $x \mapsto x_i$ of G_i, which is also a homomorphism since the gate map $x \mapsto x'$ of $G_1 \cap G_2$ is such: if e.g. more than one of u, v, w belong to $G_2 - G_1$, say the latter two, then

$$(uvw)_1 = (u_2vw)_1 = (u_2vw)' = (u'v'w') = (u_1v_1w_1)$$

because $v_1 = v'$ and $w_1 = w'$.

As to sufficiency, if some V_i includes A, then ψ is the concatenation of the gate map of G_i and ψ_i, whence it is a homomorphism. Otherwise, A is a gated amalgam of $A \cap V_1$ and $A \cap V_2$. If $\{u, v\} \subseteq V_i$ for some $i = 1, 2$, then

$$\begin{aligned} \psi(uvw) &= \psi(uvw_i) = \psi_i(uvw_i) = (\psi_i u \psi_i v \psi_i w_i) = (\psi_i u \psi_i v w_i) \\ &= (\psi_i u \psi_i v w) = (\psi u \psi v w) = (\psi u \psi v \psi w). \end{aligned}$$

The case $\{v, w\} \subseteq V_i$ is settled analogously. □

Lemma 3. *Let G be the gated amalgam of graphs G_1 and G_2. For congruences θ_1 and θ_2 of G_1 and G_2, respectively, that restrict to the same congruence of $G_1 \cap G_2$, the relation $\theta_1 \cup \theta_2$ is the* *smallest tolerance of* G *extending* θ_1 *and* θ_2, *and*

$$\theta = \theta_1 \cup \theta_2 \cup \theta_1 \circ \theta_2 \cup \theta_2 \circ \theta_1$$

is the smallest congruence of G *extending* θ_1 *and* θ_2.

Proof. To prove the first assertion, let $u(\theta_1 \cup \theta_2)x$, $v(\theta_1 \cup \theta_2)y$, $w(\theta_1 \cup \theta_2)z$, and assume that, say, u, v, x, y are from G_1 whereas w, z are from G_2. Then the gate map $'$ of $G_1 \cap G_2$ turns $w\theta_2 z$ into $w'\theta_2 z'$, so that $w'\theta_1 z'$ by hypothesis and hence

$$(uvw) = (uvw')\theta_1(xyz') = (xyz),$$

as required.

Clearly the restriction of θ to G_i equals θ_i $(i = 1, 2)$. Note that $u\theta_1 \circ \theta_2 v$ exactly when $u\theta_1 u'\theta_2 v$, or equivalently, $u\theta_1 v'\theta_2 v$.

As to transitivity of θ, assume $u\theta v\theta w$. Then $u\theta w$ follows trivially if either both u, v or both v, w belong to one of G_1 and G_2. Therefore assume that, say, u, w are in $G_1 - G_2$ and v in $G_2 - G_1$. Then $u\theta_1 v'\theta_1 w$ and hence $u\theta w$.

To prove compatibility with the imprint operation of the amalgam, assume $u\theta w$ with u from G_1, say. Then $u'\theta w'$ by Lemma 1(d) applied to θ_1 and θ_2, since either $u\theta_1 w$ or $u\theta_1 w'\theta_2 w$. If G_1 contains w and at least one of x, y, then

$$(uxy) = (ux_1y_1)\theta_1(wx_1y_1) = (wxy).$$

Else, with w in G_1 but x, y from $G_2 - G_1$, we obtain

$$(uxy) = (u'xy)\theta_2(w'xy) = (wxy).$$

Therefore it only remains to consider the case $u\theta_1 w'\theta_2 w$. By what has been shown, we infer $(uxy)\theta(w'xy)\theta(wxy)$ and hence $(uxy)\theta(wxy)$ by transitivity. Finally, $(xyu)\theta(xyw)$ is established analogously. □

From the first assertion of Lemma 3 we obtain in particular (by letting $\theta_1 = \iota_1$ and $\theta_2 = \iota_2$ be the respective "all" relations) that the two constituents of a gated amalgam are the blocks of a tolerance. This observation suggests a natural generalization of the notion of pairwise gated amalgamation: a graph G is a "tolerance" amalgam of a graph family $(G_i | i \in I)$ if the graphs $G_i (i \in I)$ constitute the blocks of a tolerance of G that covers the edge set of G. We will investigate the finest tolerance of this kind in the case of weakly median graphs; see Section 5 below.

We will apply now the preceding lemma to the situation where A is a gated set in the amalgam G of G_1 and G_2. Let A_1 and A_2 be the sets of gates of A in G_1 and G_2, respectively. Consider the relation $\theta = \theta(A)$ on G to $G_i (i = 1, 2)$ defined by

$$x\theta(A)y \Leftrightarrow (xy\psi_A y) = y \quad \text{and} \quad (yx\psi_A x) = x \text{ where } \psi_A \text{ is the gate map of } A.$$

Thus, $x\theta y$ means that $x, y, \psi_A y, \psi_A x$ constitute a *metric rectangle*, where $x, \psi_A y \in I(y, \psi_A x)$ and $y, \psi_A x \in I(x, \psi_A y)$, implying $d(x, y) = d(\psi_A x, \psi_A y)$ and $d(x, \psi_A x) = d(y, \psi_A y)$. Then $\theta = \theta_1 \cup \theta_2 \cup \theta_1 \circ \theta_2 \cup \theta_2 \circ \theta_1$ for the restrictions θ_1 and θ_2 of θ to G_1 and G_2, respectively. Indeed, if e.g. $x \in G_1 - G_2$ and $y \in G_2 - G_1$, then for the gate y' of y in $G_1 \cap G_2$ it follows that $x\theta_1 y'\theta_2 y$. Hence, under the hypothesis that θ_1 and θ_2 are congruences of G_1 and G_2, respectively, one infers from Lemma 3 that θ is a congruence of G. In this case, θ is the smallest congruence that has A as one of its blocks.

"Prefiber" has originally been employed by [53] as a synonym for "gated set", thus alluding to the fact that all "fibers" of a Cartesian product, which are the blocks of the kernels of

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the projections onto the factors, are gated sets. We have seen above that gated sets need not be congruence blocks. Even if they are, they do not necessarily participate in a subdirect representation of the graph. Take the house (Fig. 4(a)) for instance: it has only one nontrivial congruence, viz. the one with blocks $\{v, t\}, \{w, z\}, \{u\}$, so that the house is subdirectly irreducible. The gated set $A = \{u, z, t\}$ (the triangle in the house) is thus not the block of a congruence, but it has the following property, investigated by Chastand [19]: the pre-image of every vertex of A under the gate map ψ_A is a gated set in the given graph. We will refer to such gated sets as *prefibers*, thus replacing the earlier redundant use of the name. Graphs in which all gated sets are prefibers are called *fiber-complemented* in [19]. Every fiber of a direct product is a prefiber in our sense. When we consider arbitrary subdirect products, then the pre-images of single vertices under a canonical projection are trivially blocks of congruences (viz. the kernels of the projection), but they are not necessarily prefibers. Take the gated amalgam of a 6-cycle and a 4-cycle along an edge. This constitutes a subdirect product of C_6 and K_2, but one pre-image (an edge) of a vertex from the factor K_2 is not a prefiber because its gate map partitions the 6-cycle into convex but not gated parts. We suggest to call a gated set A (or the corresponding induced subgraph) a *fiber* if both the kernel $\ker\psi_A$ of the gate map ψ_A of A and the relation $\theta(A)$ are congruences. Thus, a prefiber A is a fiber exactly when there exists a congruence θ that includes $A \times A$ but intersects $\ker\psi_A$ only in the equality relation ω. Indeed, for a fiber, $\theta(A)$ is the desired congruence θ. Conversely, if $x\theta y$ for θ as described and $\psi = \psi_A$ maps x to a and y to b, then

$$\psi(xyb) = (\psi x \psi y \psi b) = (abb) = b = \psi y$$

and $(xyb)\theta(yyb) = y$, whence $(xy\psi y) = (xyb) = y$ because $\ker\psi_A \cap \theta = \omega$. Analogously, $(yx\psi x) = x$, whence $\theta \subseteq \theta(A)$. The converse inclusion is obvious since $\psi x \theta \psi y$ implies $(\psi xxy)\theta(\psi yxy)$.

If A is a fiber, then the mapping $x \mapsto ([x]\ker\psi_A, [x]\theta(A))$ is an embedding of G into the product of $G/\ker\psi_A$ (isomorphic to the subalgebra A) and $G/\theta(A)$. Indeed, if $(xy\psi_A y) = y, (yx\psi_A x) = x$, and $\psi_A x = \psi_A y$, then $x = y$ follows immediately. In this way, every nontrivial fiber leads to a proper subdirect decomposition. If the fiber A is a separating set of G, as is the case when G is amalgamated from G_1 and G_2 along A, then $G/\theta(A)$ is an amalgam of $G_1/\theta_1(A)$ and $G_2/\theta_2(A)$ (where $\theta_i(A)$ is the restriction of $\theta(A)$ to G_i for $i = 1, 2$) along the cut vertex A of $G/\theta(A)$. Hence $[x]\theta(A) \mapsto ([x_1]\theta_1(A), [x_2]\theta_2(A))$ sets up a subdirect representation of $G/\theta(A)$ by $G_1/\theta_1(A)$ and $G_2/\theta_2(A)$.

The hierarchy between the various fiber concepts from the most special “Cartesian factor” to the most general term “convex” is depicted in Fig. 6. Each arrow points from a stronger to a weaker concept, and along each link, the graph with a subset highlighted constitutes a counter-example for the converse, thus demonstrating that these notions are all different.

Note that the (finite) intersection of fibers is again a fiber. Trivially, fibers in a Cartesian product are exactly the Cartesian products of fibers in the factors. As to *fiber amalgamation*, that is, gated amalgamation along a common fiber, all fibers of the constituents stay fibers in the amalgam, and the new fibers of the amalgam are all gated amalgams of fibers in the constituents. In particular, when we start off from graphs having no nontrivial gated subgraphs, then Cartesian multiplication and gated amalgamation produces gated subgraphs that are all fibers. We summarize this discussion in the following result.

Theorem 1. *Successive fiber amalgamations from Cartesian products constitute subdirect products. Namely, let $\mathcal{K}$ be a class of graphs having only trivial gated subgraphs, and let $\mathcal{L}$*

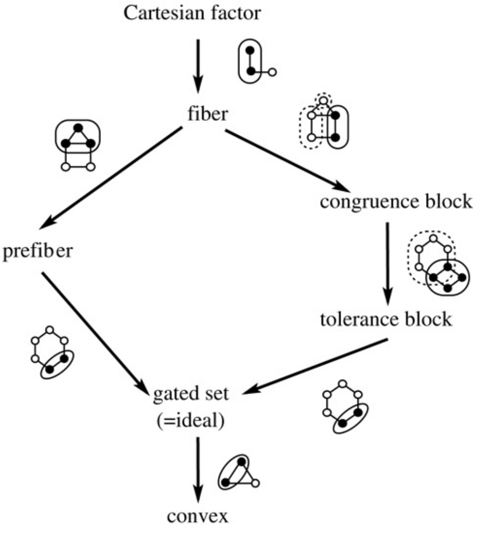

Fig. 6. Implications between fiber concepts.

be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}$. Then every graph from $\mathcal{L}$ is a (finite) subdirect product of graphs from $\mathcal{K}$ (which yield simple algebras). In particular, every finite weakly median graph is the subdirect product of prime weakly median graphs.

The class $\mathcal{L}$ in Theorem 1 obtained from the class $\mathcal{K}$ of all finite graphs having only trivial gated subgraphs has been studied by Chastand [19]: it coincides with the class of finite fiber-complemented graphs. These graphs can be characterized by a single equation in terms of the ternary operation m, which assigns to each triplet u, v, w the gate of u in $\ll v, w \gg$.

Corollary 1. *For a (finite) graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is fiber-complemented;*
(ii) *the ternary algebra (V, m) is isomorphic to the imprint algebra of a quasi-median graph;*
(iii) *the ternary algebra (V, m) satisfies Isbell's isotropy law*

$$m(m(u, v, w), x, y) = m(m(u, x, y), m(v, x, y), m(w, x, y)).$$

Proof. (i) implies (ii): By the preceding theorem, G is a subdirect product of graphs having only trivial gated subgraphs ("elementary fiber-complemented graphs" [19]). The ternary operation m on each subdirect factor is therefore the *dual discriminator* [36], that is, $m(u, v, w)$ equals v if $v = w$ and equals u otherwise. Hence m coincides with the imprint operation of the complete graph on the same vertex set. Since every congruence of G is also compatible with the operation m by Lemma 1(e), the algebra (V, m) is a subdirect product of dual discriminator algebras and hence can be interpreted as the imprint algebra of a quasi-median graph [12,43].

(ii) implies (iii): Quasi-median graphs are known to satisfy isotropy [12,43].

(iii) implies (i): Suppose that G is not fiber-complemented. Then G includes a gated set A such that the pre-image $\psi_A^{-1}(y)$ of some vertex $y \in A$ is not gated. Thus there exist vertices $v, w \in \psi_A^{-1}(y)$ and a vertex u of G such that $\psi_A u = x \neq y$ and $I(u, v) \cap I(u, w) = \{u\}$.

Clearly $u \in \psi^{-1}_{\ll x,y\gg}(x)$ and $v, w \in \psi^{-1}_{\ll x,y\gg}(y)$, whence $m(m(u, v, w), x, y) = m(u, x, y) = x$ but $m(v, x, y) = m(w, x, y) = y$, so that isotropy is violated here. □

Contrasting with the particular situation of the preceding corollary, the algebras (V, m) in general do not fit into the axiomatic framework of imprint algebras. For instance, the house (Fig. 4(a)) violates the twisted left absorption law with respect to the operation m because $m(t, w, m(w, t, v)) = m(t, w, v) = t \neq m(w, t, v)$.

Theorem 1 entails that every member G of that class $\mathcal{L}$ is isomorphic to an isometric subgraph of a Cartesian product of some graphs from the class $\mathcal{K}$. This isometric embedding may then be compared with the so-called *canonical isometric embedding* [37], described as follows. Two edges $e = xy$ and $f = uv$ of a graph $G = (V, E)$ are in Winkler's relation Θ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. This relation on the edge set E is trivially reflexive and symmetric but not necessarily transitive. Let Θ^* denote the transitive closure of Θ, and let $E_1, \ldots, E_k$ be the blocks of Θ^*. Let $G_i (i = 1, \ldots, k)$ be the graph having the connected components of the graph $(V, E - E_i)$ as its vertices, with two different components being adjacent when connected by an edge from E_i; alternatively, one can view G_i as the graph resulting from the contraction of all edges in E_i. This contraction induces a natural projection α_i from G onto G_i. Then the map $\alpha : G \to G_1 \Box \cdots \Box G_k$ defined by $\alpha v = (\alpha_1 v, \cdots, \alpha_k v)$ constitutes an isometric embedding, which is the finest isometric embedding of G into a Cartesian product (whence the name "canonical").

In general, a subdirectly irreducible apiculate graph may still have a nontrivial canonical isometric embedding into a Cartesian product, since the kernels of the projections onto the factors need not be congruences. For example, consider the 8-vertex graph G obtained from a 4-cycle by gluing four triangles along its four edges (so that the edge set of the resulting graph can be partitioned into the edge sets of the four triangles). Then the relation Θ^* on the edge set has two blocks, with either block comprising the edges of a pair of opposite triangles. Thus, the weakly median graph $K_{1,1,2} \Box K_{1,1,2}$ constitutes the Cartesian product for the canonical isometric embedding of G, although G itself has no nontrivial gated subgraphs.

In contrast to this example, all pairs of edges in a 2-connected graph G are Θ^* related whenever the graph has an ample supply of isometric odd cycles. In an odd cycle every edge is in relation Θ to its two "opposite" edges, so that Θ^* has only one block. Now, if the cycle space of G has a basis consisting of isometric odd cycles, then Θ^* has a single block; this can be proven by a straightforward induction, similarly as in the proof of [7, Lemma 3]. Such a graph G has only trivial gated subgraphs, since any isometric odd cycle sharing an edge with a gated subgraph H would be included in H, and therefore some cycle C properly intersecting H in an edge (guaranteed by 2-connectedness) would not be the modulo 2 sum of isometric odd cycles. Summarizing, we can record the following observation.

Corollary 2. *Let $\mathcal{K}_{\mathrm{odd}}$ be the class of graphs comprising K_2 and all 2-connected graphs for which the cycle spaces have bases consisting of isometric odd cycles, and let $\mathcal{L}_{\mathrm{odd}}$ be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}_{\mathrm{odd}}$. Then for every graph $G \in \mathcal{L}_{\mathrm{odd}}$ the subdirect representation from Theorem 1 constitutes a canonical isometric embedding. In particular, every finite weakly median graph has a canonical isometric embedding into a Cartesian product of prime weakly median graphs.*

A variant of Corollary 2 has been established in [19, Corollary 6.2] for pre-median graphs G, which generalize weakly median graphs: G is *pre-median* if G is weakly modular such that neither $K_{2,3}$ (Fig. 1(a)) nor the graph of Fig. 1(b) is an induced subgraph.

4. Subdirect products as retracts

A *retraction* φ of a graph $H = (W, F)$ is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \to W$ with $d(\varphi x, \varphi y) \le d(x, y)$ for all $x, y \in W$. The induced subgraph of H constituting the image of H under φ is called a *retract* of H. Retracts are isometric subgraphs, but the converse is not true in general: C_6 is an isometric subgraph but is not a retract of the 3-cube $K_2 \Box K_2 \Box K_2$. The retracts of hypercubes are precisely the median graphs [5], and more generally, the quasi-median graphs are obtained as the retracts of Hamming graphs, viz. (weak) Cartesian products of complete graphs [18,58].

A retract G of a graph H need not be a subalgebra of H (that is, of the imprint algebra of H); take, for instance, a 3-star $G = K_{1,3}$ in $H = K_{2,3}$. If, however, H is apiculate (so that imprint and apex operations coincide), then the retract G necessarily is a subalgebra of H. Gated amalgams cannot in general be obtained as retracts of Cartesian products of the constituents. The smallest counter-example is given by the gated amalgam of C_5 and K_2 along a vertex: this graph is a subdirect product of C_5 and K_2 but cannot be obtained as a retract of $C_5 \Box K_2$.

The retractions from binary products yield the key information for deciding whether an isometric subgraph $G = (V, E)$ of a Cartesian product $H = H_1 \Box \cdots \Box H_n$ is a retract. According to the elegant result of Feder (Theorem 6.35 of [35]), G is a retract of H exactly when the following two projection criteria are met:

(1) G coincides with the largest induced subgraph of H that has the same images under the projections onto all $H_i (1 \le i \le n)$ and $H_i \Box H_j (1 \le i < j \le n)$ as G; and

(2) each of these images constitutes a retract of the corresponding factor H_i or product $H_i \Box H_j$.

Under the additional requirement that G be a subdirect product of $H_1, \ldots, H_n$, condition (1) is automatically fulfilled; moreover, if the factors have no nontrivial gated subgraphs, the images of G in the binary products that remain to be checked for retractions are unions of two fibers. These observations essentially follow from purely algebraic results, the first of which is due to Bergman [15]: a subdirect product V of ternary algebras $V_1, \ldots, V_n$ satisfying the majority law $(aab) = (aba) = (baa) = a$ is uniquely determined by its images under the canonical projections onto $V_i \times V_j$ for $1 \le i < j \le n$. Namely, every point $x = (x_1, \ldots, x_n) \in V_1 \times \cdots \times V_n$ for which all coordinate pairs (x_i, x_j) have pre-images in V under the canonical projections must belong to V. The straightforward proof is by induction on n. For $n = 3$, to start with, any three pre-images $(x_1, x_2, c), (x_1, b, x_3), (a, x_2, x_3) \in V$ of points $(x_1, x_2), (x_1, x_3), (x_2, x_3)$ projected from V immediately restore

$$x = (x_1, x_2, x_3) = ((x_1 x_1 a), (x_2 b x_2), (c x_3 x_3)) \in V$$

by means of the majority law. The second result (also formulated here only for ternary algebras) generalizes an observation of Fried and Pixley [36] on dual discriminator algebras.

Lemma 4. *Let a ternary algebra V be a subdirect product of two algebras V_1 and V_2 that satisfy the majority law and have only trivial ideals. Then either the factorization is trivial ($V \cong V_1$ or $V \cong V_2$), or $V = V_1 \times V_2$ is the whole direct product, or V is the union of two ideals (fibers) of the form $\{v_1\} \times V_2$ and $V_1 \times \{v_2\}$.*

Proof. If $(u_1, u_2), (v_1, v_2), (v_1, w_2) \in V$, then $(v_1, (u_2 v_2 w_2)) = ((u_1 v_1 v_1), (u_2 v_2 w_2)) \in V$. Hence, the pre-image $\{v_1\} \times W_2$ of v_1 (for some $\emptyset \ne W_2 \subseteq V_2$) under the (first) projection of V onto V_1 is an ideal of V, whence it is either a singleton or equals V_2 by the hypothesis. The analogous statement holds with respect to the second projection. It is then easy to see that only three cases can occur for the pre-images of single points in V_1 and V_2: (i) they are all singletons, (ii) they are equal to the associated fibers of $V_1 \times V_2$, and (iii) they are singletons except for two fibers $\{v_1\} \times V_2$ and $V_1 \times \{v_2\}$. This proves the assertion of the lemma. □

Returning to graphs, we thus have the following result that specifies Feder's theorem in the algebraic scenario.

Corollary 3. *Every subdirect product of graphs $H_1, \ldots, H_n$ that have no nontrivial gated subgraphs is a retract of $H_1 \Box \cdots \Box H_n$ if and only if for all $1 \le i < j \le n$ the gated amalgam of two copies of H_i and H_j along a single common vertex is a retract of $H_i \Box H_j$.*

In general, it is difficult to decide whether the gated amalgam of two graphs along a vertex is a retract of the two graphs: even when the second graph is fixed as K_2, this decision problem is as difficult as the SAT problem and hence NP-complete [36, Lemma 6.32]. As we will show below, these retraction questions are closely related with a combing property of graphs which comes from the geometric theory of groups [32]. Let b be a distinguished vertex ("base point") of a graph G and let k be an integer. Two paths $P(x,b)$, $P(y,b)$ in G connecting two vertices x, y to b are called *k-fellow travelers* if $d(x',y') \le k$ holds for each pair of vertices $x' \in P(x,b)$, $y' \in P(y,b)$ with $d(x,x') = d(y,y')$. A *geodesic k-combing* of G with respect to the base point b comprises shortest paths $P(x,b)$ between b and all vertices x such that $P(u,b)$ and $P(v,b)$ are k-fellow travelers for any edge uv of G. One can select the combing paths so that their union is a spanning tree T_b of G that is rooted at b and preserves the distances from b to all vertices. The neighbor $f(x)$ of x in the unique path of T_b connecting x with the root will be called the *father* of x. A geodesic 1-combing of G with respect to b (also referred to as a mooring in G onto $\{b\}$; see [20]) thus amounts to a tree T_b preserving the distances to the root b such that if u and v are adjacent in G then $f(u)$ and $f(v)$ either coincide or are adjacent in G. The k-houses for $k \ge 1$ admit geodesic 1-combings with respect to all base points. In [21, 25] it is noticed (using [24]) that for bridged graphs every spanning tree returned by Breadth-First-Search (BFS) starting from b provides a geodesic 1-combing. Trivially the same holds for hyperoctahedra and 5-wheels.

For any two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with base points a and b, respectively, we denote by $G_1 \dotplus_{a,b} G_2$ the gated amalgam of the two fibers of $G_1 \Box G_2$ that share the vertex (a,b), i.e., the subgraph of $G_1 \Box G_2$ induced by $(\{a\} \times V_2) \cup (V_1 \times \{b\})$. The equivalence of the first two conditions in the following lemma is due to [22, Theorem 2.3.1].

Lemma 5. *For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with at least two vertices the following conditions are equivalent:*

(i) *$G_1 \dotplus_{a,b} G_2$ is a retract of $G_1 \Box G_2$ for each vertex (a,b) of $G_1 \Box G_2$;*
(ii) *$G_1 \dotplus_{a,0} K_2$ and $G_2 \dotplus_{b,0} K_2$ are retracts of $G_1 \Box K_2$ and $G_2 \Box K_2$, respectively, for all vertices a of G_1, b of G_2, and vertex 0 of K_2;*
(iii) *G_1 and G_2 each have geodesic 1-combings with respect to all base points.*

Proof. (i) implies (ii): Take a neighbor c of b in G_2, so that we can regard the edge bc as a copy of K_2. Then the given retraction φ from $G_1 \Box G_2$ to $G_1 \dotplus_{a,b} G_2$ restricts to a retraction from $G_1 \Box K_2$ to $G_1 \dotplus_{a,b} K_2$ because the convex hull of $G_1 \dotplus_{a,b} K_2$ in $G_1 \Box G_2$ equals $G_1 \Box K_2$, which is therefore mapped into itself by φ.

(ii) implies (iii): Let K_2 have the vertices 0 and 1. Since the retraction φ from $G \Box K_2$ to $G_1 \dotplus_{a,0} K_2$ maps $(V_1 - \{a\}) \times \{1\}$ into $V_1 \times \{0\}$ we can define a "father" map $f_1 : V_1 \to V_1$ that preserves or collapses the edges of G_1 via

$$f_1(a) = a \quad \text{and} \quad (f_1(x), 0) = \varphi(x, 1) \quad \text{for } x \neq a.$$

Then, f_1 maps the base point a to itself and any other vertex x of G_1 to a neighbor $f_1(x)$ of x in $I(a, x)$. Let the spanning tree T_a of G_1 consist of all edges $f_1(x)x$ for $x \in V_1 - \{a\}$. This tree obviously preserves the distances from a to all vertices, and moreover, the paths have the 1-fellow property. Hence the paths in T_a emanating from a provide a geodesic 1-combing of G_1. Analogously, one obtains a geodesic 1-combing of G_2.

(iii) implies (i): Let f_1 and f_2 denote the two father maps in the spanning trees T_a and T_b that yield the geodesic 1-combings of G_1 and G_2. We construct a retraction φ from $G_1 \Box G_2$ to $G_1 \dot{+}_{a,b} G_2$ as follows. For each vertex (u, v) of $G_1 \Box G_2$ take the smallest $k \geq 0$ such that either $f_1^k(u) = a$ or $f_2^k(v) = b$; then $\varphi(u, v)$ is set to $(f_1^k(u), f_2^k(v))$. In other words, we repeatedly apply the father map pair (f_1, f_2) to (u, v) until we first reach one of the two fibers containing (a, b). By definition, φ is an idempotent map to $G_1 \dot{+}_{a,b} G_2$. To show that φ is nonexpansive, that is, φ preserves or collapses edges, assume without loss of generality that vw is an edge of G_2 with $d(b, v) \leq d(b, w)$. Let $\varphi(u, v) = (f_1^k(u), f_2^k(v))$ with k minimal. Since both father maps preserve or collapse edges, the only nontrivial case to check is when $f_1^k(u) \neq a$ and $f_2^k(v) = b \neq f_2^k(w)$. As vw is an edge, it follows that $f_2^{k+1}(w) = b$, whence $\varphi(u, w) = (f_1^{k+1}(u), b)$ is adjacent to $\varphi(u, v)$. □

From this lemma and Lemma 6.32 of [35] we conclude that recognizing graphs which have a geodesic 1-combing is NP-complete. For some classes of graphs the lemma provides us with geodesic 1-combings that are not simply constructed via BFS. For instance, the class of Helly graphs (alias absolute retracts of reflexive graphs [13]) is trivially closed under gated amalgamations along vertices. Therefore the existence of geodesic 1-combings is guaranteed for Helly graphs.

Corollary 4 (*[20]*). *Let $\mathcal{K}_\Delta$ be the class comprising all subhyperoctahedra, 2-connected bridged graphs, and 2-connected Helly graphs, and let $\mathcal{L}_\Delta$ be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}_\Delta$. Then every graph from $\mathcal{L}_\Delta$ is a retract of a Cartesian product of graphs from $\mathcal{K}_\Delta$. In particular, every finite weakly median graph is a retract of a Cartesian product of prime weakly median graphs, and vice versa.*

5. Subdirect representation of infinite weakly median graphs

In establishing the subdirect representation for finite fiber-complemented graphs (Theorem 1), we have not yet employed the full information provided in [19], which extends to the infinite case as well.

Lemma 6 (*[19]*). *Let S be the smallest gated subgraph that contains the edge vw of a fiber-complemented graph G. Then the blocks $W_s = \psi_S^{-1}(s) (s \in S)$ of the kernel of the corresponding gate map ψ_S contain isomorphic gated subgraphs $U_s (s \in S)$ such that their union induces a gated subgraph $H \cong S \Box U$ (where U may be any U_s), which together with all blocks $W_s (s \in S)$ covers the edge set of G. If H is not all of G, then G is a gated amalgam of $G - (W_s - U_s)$ and W_s along U_s for some $s \in S$ with $U_s \neq W_s$.*

Proof. The prime gated subgraphs of G are precisely the smallest gated subgraphs generated from single edges of G, by virtue of [19, Lemmas 4.4 and 4.8]. The Cartesian decomposition

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$H \cong S \Box U$ follows from [19, Theorem 5.2], and the amalgamation of $G - (W_s - U_s)$ and W_s is established in the proof of [19, Theorem 5.4]. □

For the particular case of weakly median graphs, the preceding lemma can also be inferred from [7]. Namely, by [7, Lemmas 3 and 5], S either comprises the edge vw or is 2-connected and null-homotopic. It is explicitly shown in the proofs of [7, Lemma 8 and 10] that all U_s are gated and each vertex of H belongs to an isomorphic copy of S which is a transversal for the subgraphs $U_s (s \in S)$. This immediately implies that for any vertex x outside H the gate of x in U_t, where t is the gate of x in S, serves as the gate of x in H.

Lemma 7. *The smallest gated subgraph S that contains the edge vw of a fiber-complemented graph G is a prime subgraph giving rise to a minimal nontrivial tolerance/congruence $\theta = \theta(S)$, which has S as one of its blocks. Then* $\ker\psi_S$ *is a congruence, which equals the pseudocomplement θ^* of θ in the tolerance lattice of G, that is,* $\ker\psi_S$ *is the largest tolerance intersecting θ in the equality relation. Thus, G/θ^* is isomorphic to S.*

Proof. S is a prime graph according to [19, Lemma 4.8]. In view of the product representation $H \cong S \Box U$ described in the previous lemma, S is a block of the canonical congruence $\theta_H(S) = \ker\psi_U|_H$ of the Cartesian product H. We can trivially extend θ_H to the required congruence of G so that $\theta - \theta_H$ is the equality relation on $G - H$. Clearly, $\theta_H(S)$ is a minimal nontrivial congruence, and hence so is θ.

The smallest gated subgraph T generated from any edge xy of the subgraph $W_s = \psi_S^{-1}(s)$, where $s \in S$, either coincides with S (if xy belongs to $U_s \subseteq H$) or is included in W_s such that $|T \cap U_s| \leq 1$ (cf. [19, Lemma 4.9]). Therefore $\theta(T) \subseteq \ker\psi_S$, and hence $\ker\psi_S$ equals the union of all (finitary) relational products of minimal congruences that are contained in $\ker\psi_S$. This qualifies $\ker\psi_S$ as a congruence, which intersects $\theta(S)$ in the equality relation. Since every minimal nontrivial congruence different from $\theta(S)$ is contained in $\ker\psi_S$, it follows that $\ker\psi_S = \theta(S)^*$. □

Algebraically, fiber amalgamation is determined by a tolerance with exactly two blocks that is not transitive (i.e. with two intersecting blocks). Thus, successive fiber amalgamation is manifest in a particular tolerance with several blocks, by virtue of Lemma 3. The smallest tolerance β of this kind will then testify to all possible decompositions via fiber amalgamation, just as in the particular case of median graphs [3]. Its blocks are isomorphic to weak Cartesian products of prime constituents; see the next theorem.

Let I be an infinite index set, and let $F_i = (V_i, E_i)$, $i \in I$, be any graphs with at least two vertices. A *weak Cartesian product* of this family of graphs is any connected component H of the "infinitary Cartesian product" with vertex set $\prod_{i \in I} V_i$ and edges xy for which

$$x_j y_j \in E_j \quad \text{for some } j \in I \quad \text{and} \quad x_i = y_i \quad \text{for all } i \in I - \{j\}.$$

Thus, each graph F_i forms a Cartesian factor of H, and H itself constitutes a directed union of (finitary) Cartesian products. Namely, select a base point b of H and consider the fibers $F_i(b)$ of the infinitary Cartesian product that correspond to F_i and contain b. Then the component $F = F(b)$ that contains b comprises those vertices which differ from b in only finitely many coordinates. F is thus the directed union of the convex hulls of all finite subfamilies of $(F_i(b)|i \in I)$, which constitute (finitary) Cartesian products. Algebraically, the imprint algebra of F is a weak direct product (sensu [38, p. 139]) of the imprint algebras of the family $(F_i|i \in I)$. Bergman's theorem for finitary direct products of ternary algebras (satisfying the majority law) immediately extends to weak direct products by virtue of a trivial induction. Then, in particular, a nonempty subset W of the imprint algebra of F is a subalgebra if and only if its projection on each binary product $V_i \times V_j$ ($i \neq j$ from I) constitutes a subalgebra; moreover, it is uniquely determined by these projections. Therefore the congruences $\theta_i = \theta(F_i)(i \in I)$ of H, which are complementary in H to the canonical congruences, permute in pairs, that is, $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ for $i \neq j$. This congruence permutation property is characteristic of the weak Cartesian product H among all subgraphs G of H that constitute subdirect products of the graphs $F_i (i \in I)$.

Theorem 2. *Every fiber-complemented graph G is a subdirect product of the prime fiber-complemented graphs G/θ^* associated with the minimal nontrivial congruences θ of G. The smallest tolerance β of G that covers the edge set of G equals the intersection of all tolerances with two intersecting blocks and can also be expressed as the union of all relational products of pairwise commuting minimal congruences. Hence the blocks of β are the maximal gated subgraphs of G isomorphic to weak Cartesian products of prime fiber-complemented graphs.*

Proof. The intersection of all $\theta(S)^*$, with S running through the gated subgraphs generated from single edges, equals the equality relation ω; for otherwise, some minimal nontrivial congruence $\theta(T)$ would be contained in this intersection and hence in its pseudocomplement $\theta(T)^*$, which is absurd. Therefore G is a subdirect product of those graphs S, by Birkhoff's theorem (see [38, Section 20, Theorem 1]).

First note that for any two tolerances ξ_1 and ξ_2 with $\xi_1 \cap \xi_2 = \omega$ the relation $(\xi_1 \circ \xi_2) \cap (\xi_2 \circ \xi_1)$ is the smallest tolerance containing both ξ_1 and ξ_2 because the imprint algebra satisfies the majority law [28, Lemma 3.8]. Hence, if ξ_1 and ξ_2 commute, this tolerance equals $\xi_1 \circ \xi_2 = \xi_2 \circ \xi_1$. A straightforward induction shows that for any tolerances $\xi_1, \dots, \xi_n$ that pairwise commute and intersect in ω the relational product $\xi_1 \circ \cdots \circ \xi_n$ is the smallest tolerance containing them since it is easy to see that $\xi_1 \circ \cdots \circ \xi_{n-1} \cap \xi_n = \omega$. Applied to the minimal nontrivial congruences of G, this yields that the requirement to cover all edges forces β to contain all relational products $\theta_1 \circ \cdots \circ \theta_n (n \geq 1)$ of minimal nontrivial congruences θ_i that commute in pairs.

Now, assume that some pair (u, x) of vertices is not a member of any of those relational products. Take a shortest path $u_0 = u, u_1, \dots, u_{k-1}, u_k = x$ in G (necessarily of length $k \geq 2$) and consider the (not necessarily distinct) gated subgraphs S_i generated from the edges $u_i u_{i+1}$ ($i = 0, \dots, k-1$), respectively. For the associated congruences $\theta_i = \theta(S_i)$, choose $i < j$ with $j - i$ minimal such that θ_i and θ_j do not commute. Then

$$u\theta_0 \circ \cdots \circ \theta_{i-1} \circ \theta_{i+1} \circ \cdots \circ \theta_{j-1} v \theta_i w \theta_j u_{j+1} \theta_{j+1} \circ \cdots \circ \theta_k x$$

for some vertices v and w from $I(u, x)$. Since θ_i and θ_j do not commute, $I(v, w)$ and $I(w, u_{j+1})$ generate gated subgraphs T_i and T_j, respectively, such that $\theta(T_i) = \theta(S_i), \theta(T_j) = \theta(S_j)$, and $T_i \cap T_j = \{w\}$. Thus, when T_i plays the role of S and w the role of s in Lemma 6, we infer that G is a gated amalgam of $G - (W_s - U_s)$ and W_s along U_s. Since u_{j+1} is contained in $W_s - U_s$, so is x, whereas u belongs to W_v and hence to $G - W_s$. Therefore (u, x) is not a member of the tolerance having $G - (W_s - U_s)$ and W_s as its two blocks. This proves that β, being sandwiched by the union of all (finitary) relational products of commuting families of minimal nontrivial congruences and the intersection of all tolerances with exactly two blocks that intersect, must coincide with both relations, thus establishing the theorem. □

From Corollary 4 and the preceding theorem we can readily derive the retract theorem of Chastand [20].

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Corollary 5 (*[20]*). *Let $(G_i|i \in I)$ be the family of prime constituents of a fiber-complemented graph G such that every $G_i (i \in I)$ has a geodesic 1-combing. Then G is a retract of a weak Cartesian product H of $(G_i|i \in I)$.*

Proof. For each pair $i \neq j$ from I there is a retraction φ_{ij} from H to the pre-image G_{ij} of the canonical projection of G onto $G_i \Box G_j$, by virtue of the hypothesis and Lemma 5. The intersection of all subgraphs G_{ij} of H thus includes G and has the same projection to each $G_i \Box G_j$ as G, whence it must equal G by Bergman's theorem. Since every vertex x of G has finite distance to a fixed vertex b of G, there are only finitely many maps among the retraction maps φ_{ij} $(i \neq j)$ for which the restriction $\varphi_{ij}|_{I(b,x)}$ is not the identity map. Hence we can define the infinitary concatenation φ of the family $(\varphi_{ij}|i \neq j \text{ from } I)$ with respect to some (well-) ordering $\prec$ of $I \times I$ by letting $\varphi x = \varphi_{i_n j_n} \circ \cdots \circ \varphi_{i_1 j_1} x$ where $(i_1, j_1) \prec \cdots \prec (i_n, j_n)$ constitute the index pairs for which $\varphi_{i_\nu j_\nu}$ is not the identity on $I(b, x)$. □

In the infinite case, one is interested in conditions of local finiteness (see [45] for an extensive treatment), which involve finite subsets W of the vertex set V and their convex hulls, for instance.

Corollary 6. *The convex hull of a finite set F of vertices in a fiber-complemented graph G is the subdirect product of finitely many prime fiber-complemented graphs.*

Proof. For every vertex pair u, x there are at most $d(u, x)$ distinct minimal nontrivial congruences θ such that any shortest path from u to x passes through an edge vw with $v\theta w$. This follows from Lemma 7. Hence for all but finitely many minimal nontrivial congruences θ the vertices in F are all congruent modulo the pseudocomplement θ^*. □

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