The algebra of metric betweenness II: geometry and equational characterization of weakly median graphs

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THE ALGEBRA OF METRIC BETWEENNESS II: GEOMETRY AND EQUATIONAL CHARACTERIZATION OF WEAKLY MEDIAN GRAPHS

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Abstract. We elaborate on the convexity properties of (not necessarily finite) weakly median graphs and their prime constituents in order to establish a number of equations in terms of the weakly median operation. Then the weakly median graphs can be identified with the discrete members of an equational class of ternary algebras satisfying five independent axioms on two to four points. This demonstrates that the median algebras featured by Avann and Sholander half a century ago and, more generally, Isbell's isotropic media can be generalized much further, without losing the close ties with graphs.

1. Introduction

Weakly median graphs are defined in terms of certain intersection properties of the sets (intervals) that comprise all shortest paths between any pair of vertices. What makes weakly median graphs so appealing is that they admit a decomposition scheme into a number of nontrivial prime constituents. These encompass subhyperoctahedra, the 5-wheel, and the graphs embeddable in the plane such that all inner faces are triangles and all inner vertices have degrees larger than 5. The operations participating in this composition scheme are either (1) weak Cartesian multiplication and gated amalgamation, or (2) weak Cartesian multiplication and retraction, or (3) subdirect multiplication [2].

Since the latter composition can be formulated within in a purely algebraic framework involving the imprint operation, the question arises whether weakly median graphs can be described as ternary algebras satisfying an additional discreteness condition. Among graphs, the weakly median property can well be expressed in term of equations, but the equations exhibited so far [2] do not suffice to guarantee that an arbitrary discrete algebra fulfilling those equations can actually be derived from a graph. The additional constraint of graphicity is not a problem in the more specific context of quasi-median algebras [8] because the few defining equations are then strong enough to ensure the required graph realization. The (planar) bridged prime constituents of weakly median graphs (plane triangulations) therefore need more attention in regard to valid equations. In particular, we will have to study the geometric properties of convex hulls (of 4-point sets) in more detail.

For all basic definitions and results we refer to part I [2]. In particular, we use the abbreviations (T) and (Q) for the triangle and quadrangle conditions as well as their sharper versions (T!) and (Q!) that hold in weakly median graphs. Since in part I we have already introduced equations (A1) through (A7) for ternary operations, we continue to number new axioms here beginning with (A8).

The present paper is then organized as follows. Section 2 confirms that properties of the imprint operation (expressed by equations) in weakly median graphs need only be proven in the finite case because finitely generated weakly median graphs are necessarily finite. In Section 3, the convex hulls of metric triangles in weakly median graphs are determined. In the particular case of plane triangulations, the sides of any metric triangle extend to separating convex paths that partition the planar graph into regions that are relevant for locating point and interval shadows involving the corners of the metric triangle. This is a basic tool for verifying a number of equations in weakly median graphs that reflect their geometric structure. The equations considered in Section 4 capture a number of graph properties enjoyed by weakly median graphs. Several combinations of these equations are then characteristic for this class of graphs, and a number of subclasses (such as the class of quasi-median graphs) can be described by some stronger equations (Section 5). The final section presents the main result (Theorem 2), by which weakly median graphs can be identified with discrete ternary algebras satisfying one of three sets of independent equations in four variables.

2. Join-hull commutativity and finite generation

The interval between two vertices, and thus their convex hull, in an infinite hyperoctahedron, for example, is infinite. But if such obstructions do not occur, then the convex hull of a finite set in an infinite weakly median graph is generated by finitely many finite intervals, as we will see next. A graph $G = (V, E)$ is called a *Peano graph* if its intervals satisfy the following property:

Peano axiom: for any vertices $u, v, w \in V$, $x \in I(u, v)$, and $y \in I(w, x)$, there exists a vertex $z \in I(v, w)$ such that $y \in I(u, z)$.

One can show that Peano graphs are exactly the graphs in which the convexity is *join-hull commutative* (cf. [15]), that is $\mathrm{conv}(A \cup \{x\}) = \cup_{z \in A} I(x, z)$ holds for every convex set A and vertex x. It was shown in [11] that all weakly median graphs fulfill the Peano axiom. In view of the subdirect representation available now, we can give a somewhat shorter proof for this result.

Proposition 1 ([11])**.** *Weakly median graphs are Peano graphs.*

Proof. Since gated amalgamation and Cartesian multiplication preserve the Peano property (cf. [4] and [15, Theorem 5.14]), we actually need to verify the Peano axiom only in the prime case by virtue of [2, Corollary 6]. Since all subhyperoctahedra trivially satisfy this property, we can assume that the (prime) weakly median graph G under consideration is C_4- and K_4-free.

Let $x = v_0, v_1, \ldots, v_k = v$ be a shortest path connecting x and v. The Peano property for u, v, w, x, y then follows from that of u, v_i, w, v_{i-1}, y for $i = 1, \ldots, k$. Indeed, this yields a sequence $z_1, \ldots, z_k$ of vertices with $z_{i-1} \in I(u, z_i)$ for $i = 1, \ldots, k$ (where $z_0 = y$), so that $z = z_k$ is the required vertex. Similarly, given a shortest path $x = x_0, x_1, \ldots, x_j = y$, the Peano axiom for u, v, w, x, y is settled once we have shown it for $u, z_{i-1}, w, x_{i-1}, x_i$ by producing the necessary vertex z_i for all $i = 1, \ldots, j$ (where $z_0 = v$). Therefore we can assume that x is adjacent to v and y.

If $d(v, w) > d(x, w)$, then we can choose $z = y$ in order to fulfill the Peano axiom. If $d(v, w) < d(x, w)$, then v, x, y form a triangle because G satisfies (Q) and is C_4-free. Then, by (T), v and y have a common neighbor v' in $I(v, w)$. If $d(u, x) = d(u, y)$, then we can choose $z = v$. Therefore assume $d(u, v) = d(u, y)$. Then $d(u, v') > d(u, v)$ so that $z = v'$ does the job, because otherwise either (T!) is violated (if $d(u, v') < d(u, v)$) or we would have a triangle (formed by v, v', y) equidistant to u, which must have a common neighbor one step closer to u, thus producing a forbidden K_4.

Therefore only the case $d(v, w) = d(x, w)$ remains to be investigated. If y is adjacent to v, then $z = v$ can be chosen. Otherwise, by (T), v and x have a common neighbor v' in $I(v, w)$ different from y. As G is C_4-free, v' and y are adjacent, which by (T) then have a common neighbor v'' in $I(v', w)$. Now, if $d(u, y) = d(u, x)$, then $d(u, v) = d(u, v')$ and thus $z = v'$ is the desired vertex, because otherwise x, y, v' would form a triangle equidistant to u. Hence we can assume $d(u, y) = d(u, v)$. Then x is the median of u, v, y, and therefore v' as a common neighbor of v, y, and x must be at distance $d(u, v)$ from u as well. If $d(u, v'') \leq d(u, v)$, then we get a conflict with (T!) or a forbidden triangle equidistant to u. Therefore $d(u, v'') > d(u, y)$, whence $z = v''$ concludes the proof. $\square$

Lemma 1. *All intervals in a weakly median graph G are finite when G does not contain an induced infinite $K_{1,\ldots,2}$ (a countable complete graph minus one edge).*

Proof. When some infinite intervals occur, take one, $I(u, v)$, for which $d(u, v)$ is as small as possible. If $d(u, v) = 2$, then $I(u, v)$ constitutes a subhyperoctahedron and hence G contains $K_{1,\ldots,2}$ as an induced subgraph. Therefore assume $d(u, v) \geq 3$. If the set of neighbors of v in $I(u, v)$ was finite, then $I(u, v)$ would be the union of finitely many finite (sub)intervals. Therefore v has an infinite number of different neighbors $w_0, w_1, w_2, \ldots$ at distance $d(u, v) - 1$ to u. By weak modularity w_0 has a common neighbor x_i with each w_i ($i \geq 1$). Since $I(u, w_0)$ contains all x_i ($i \geq 1$) but must be finite by hypothesis, at least one interval $I(x_j, v)$ for some $j \geq 1$ is infinite, contrary to the minimality of $d(u, v) \geq 3$. □

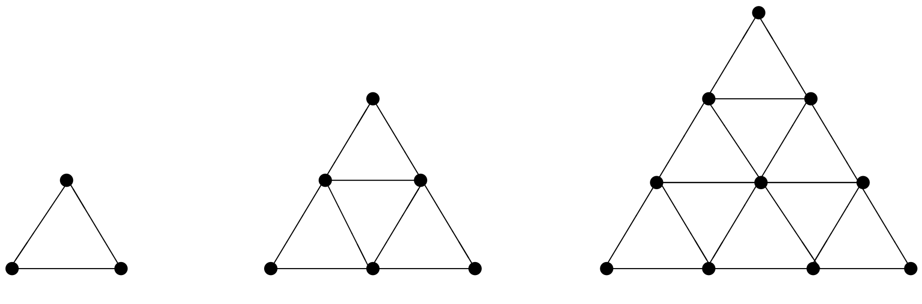
Proposition 2. *The subalgebra S generated by a finite set X in a weakly median graph G is contained in a finite (weakly median) induced subgraph of G that constitutes a subalgebra of the imprint algebra of G.*

Proof. According to [2, Corollary 6], there are only finitely many nontrivial projections of X into the prime factors in a subdirect representation of G. In all factors that are not infinite subhyperoctahedra the convex hulls of the projected vertices from X are finite by Lemma 1. Every finite subset (with at least two vertices) of an infinite subhyperoctahedron can be connected by adding at most one vertex, so that $K_{1,2}, C_4$, or a finite subhyperoctahedron arises. Taking the Cartesian product of the former finite convex hulls and the latter finite graphs results in a finite weakly median graph that is a subalgebra containing X and hence S. □

By this proposition, every equation that holds in the imprint algebras of all finite weakly median graphs is also true for all infinite weakly median graphs.

3. Deltoids

The *triangular grid* is the tessellation of the plane into equilateral triangles of equal (unit) size. The convex hull Δ of a metric triangle xyz of size $k \geq 0$ in the triangular grid either is a single vertex (if $k = 0$) or constitutes an equilateral triangle of size k that is subdivided into unit triangles by lines parallel to its sides. We refer to such a graph as a *k-deltoid* with *corners* x, y, z and *sides* $I(x, y), I(x, z), I(y, z)$ (see Fig. 1 for $k = 1, 2, 3$). The interior Δ° of the deltoid Δ is defined as Δ minus the three sides of Δ; the interior $I^\circ(a, b)$ of a side $I(a, b)$ of Δ (where $a, b \in \{x, y, z\}$) equals $I(a, b) - \{a, b\}$.



(a) $k = 1$ (K_3) (b) $k = 2$ (sun) (c) $k = 3$

FIGURE 1. The first three k-deltoids

Proposition 3. *An apiculate graph G is weakly median if and only if the convex hull* $\operatorname{conv}(u, v, w)$ *of any metric triangle uvw is a Cartesian product of deltoids.*

Proof. By [2, Corollary 6], $\operatorname{conv}(u, v, w)$ is the subdirect product of finitely many prime weakly median graphs. Since metric triangles in gated amalgams must belong to one of the constituents, it follows [2, Theorem 2] that $\operatorname{conv}(u, v, w)$ is a Cartesian product of prime graphs. Each of these prime factors is then the convex hull of a metric triangle. Since metric triangles in 5-wheels and induced subgraphs of hyperoctahedra are just triangles and thus 1-deltoids, we can henceforth assume that G is a prime weakly median bridged graph. Note that the metric triangle uvw in G is necessarily equilateral such that all vertices in $I(v, w)$ have the same distance to u, by virtue of weak modularity [6, 10]. If $I(v, w)$ was not a path, then it would include two adjacent vertices x, y equidistant to v because G is bridged. Then, by applying (T) three times (to x, y with respect to u, v, w), we obtain three distinct common neighbors of x and y: two in $I(v, w)$ and one in $I(u, x)$. This necessarily yields either an induced $K_{1,1,3}$ or a K_4, which, however, are forbidden in a prime weakly median bridged graph. Therefore the three intervals $I(u, v), I(v, w), I(w, u)$ are convex paths.

To complete the proof, we proceed by induction on the size $k \geq 2$ of uvw. Let $v = x_0, x_1, \ldots, x_{k-1}$, $x_k = w$ be the convex path constituting $I(v, w)$. Applying (T) to each pair x_{i-1}, x_i with respect to u, we obtain vertices $y_1, \ldots, y_k$ at distance $k - 1$ to u such that y_i is adjacent to x_{i-1}, x_i for $i = 1, \ldots, k$. Since $I(v, w)$ is a convex path and G is bridged, the vertices $y_1, \ldots, y_k$ are different and induce a convex path with $d(y_1, y_k) = k - 1$. Hence uy_1y_k is a metric triangle of size $k - 1$. By the induction hypothesis, $\operatorname{conv}(u, y_1, y_k)$ is a $(k - 1)$-deltoid. This together with $I(u, v)$ induces a k-deltoid Δ. For $k = 2$, suppose by way of contradiction that $I(u, x_1)$ contains a third common neighbor z of u and x_1 besides y_1 and y_2. Then z must be adjacent to y_1 and y_2 because G is bridged, thus producing a forbidden K_4. Hence assume $k \geq 3$. Let t be the neighbor of u in $I(u, v)$. Then, by exchanging the roles of u and v, we obtain another convex $(k - 1)$-deltoid with corners v, t, and x_{k-1}. This is necessarily induced by the convex path $I(v, x_{k-1})$ together with the $(k-2)$-deltoid with corners y_1, t, and y_{k-1}. An analogous statement can be made when u and w are interchanged. We conclude that Δ is the union of three convex $(k-1)$-deltoids (each containing exactly one of u, v, w). Therefore each pair of vertices at distance 2 in Δ belongs to a convex $(k - 1)$-deltoid, so that by [3, Lemma 1] Δ is convex in this case. This establishes the "only if" part of the proposition.

To establish the converse, observe that when the convex hull of any metric triangle (uvw) in G is a Cartesian product of deltoids then all vertices of $I(v, w)$ are equidistant to u. This property entails that G is weakly modular by [10, Theorem 2]. Since G is apiculate, G is weakly median. $\square$

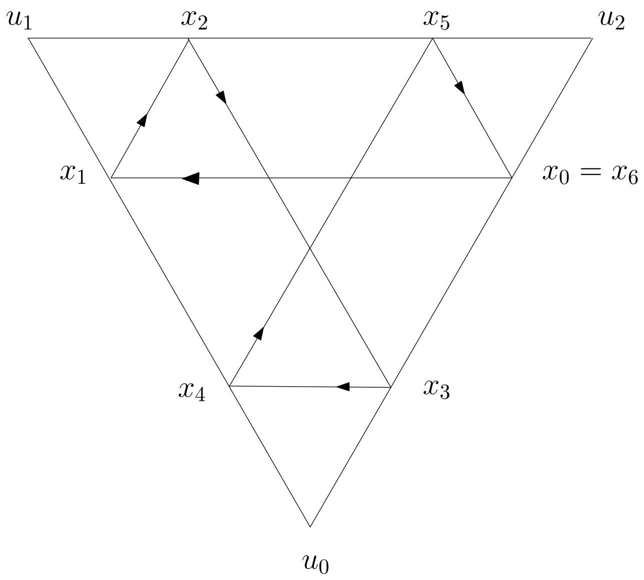


FIGURE 2. The billiard law in a deltoid

As subalgebras, deltoids are generated by their three corners plus one neighbor of a corner. To see this, consider a k-deltoid Δ ($k \geq 2$) with corners u_0, u_1, and u_2. Then for any vertex $x_0 \in I(u_0, u_2)$ with $d(x_0, u_2) = j \leq k/2$, say, one can iteratively define the "billiard sequence"

$$x_{i+1} = (u_{i+1} u_i x_i) \text{ for } 0 \leq i \leq 5$$

where the indices of u are read modulo 3. Then

$$d(x_1, u_1) = d(x_2, u_1) = d(x_3, u_0) = d(x_4, u_0) = d(x_5, u_2) = d(x_6, u_2) = j,$$

whence $x_6 = x_0$. We refer to this property as to the *billiard law;* see Fig. 2. For $j = 1$, in particular, Δ is covered by the three $(k-1)$-deltoids with corner sets $\{u_0, x_0, x_1\}, \{u_1, x_4, x_5\}$, and $\{u_2, x_2, x_3\}$, respectively. In each of these $(k-1)$-deltoids, some vertex from $\{x_0, \ldots, x_5\}$ is adjacent to a corner and thus can start a billiard sequence within this $(k-1)$-deltoid. Thus a trivial induction shows that all vertices of Δ are generated by u_0, u_1, u_2, and x_0. Since deltoids can be arbitrarily large, the distance between two generators of a 4-generated weakly median graph G cannot be bounded from above, quite in contrast to 4-generated quasi-median graphs, which have no more than 868 vertices [5, 14]. Let us call a ternary algebra a *weakly median algebra* if it satisfies all equations true for finite weakly median graphs. The free 3-generated weakly median algebra coincides with the "free taut medium on 3 generators" [12, p. 331] and is represented by the familiar 6-vertex graph of [2, Fig. 3(a)]. In view of Proposition 3 and the 4-generation of deltoids we conclude that the free 4-generated weakly median algebra is infinite and is not the imprint algebra of a graph (because it has infinite bounded chains).

According to [3], prime weakly median bridged finite graphs $G = (V, E)$ are exactly represented by the plane triangulations in which all inner vertices have degree larger than 5. The neighborhood $N(x) = \{y \in V | \ y \text{ is adjacent to } x\}$ of every inner vertex induces a cycle, whereas the vertex incident with the external face have a path neighborhood. By a *line* of G (thus embedded in the plane) we mean the vertex set of a convex path whose end vertices both belong to the external face of G. Every convex path extends to a line. To show this, let L be a convex path $u = v_0, v_1, \ldots, v_k = v$ ($k \geq 1$) such that the end vertex v is an inner vertex of the plane graph G. Since the degree of v is then at least 6, there exists a vertex w adjacent to v but not to v_{k-1} such that v is the only common neighbor of v_{k-1} and w. Hence, by [3, Lemma 1], L plus w induces a convex path. This shows that there is an ample supply of lines because every edge can be extended to a line. A line is either separating, that is, its removal disconnects G or it is part of the boundary of the external face. The *border* L of a halfspace A of G comprises the vertices $x \in A$ for which the neighborhood $N(x)$ intersects the complementary halfspace $A' = V - A$. Then $L' = N(L) \cap A'$ is the border of A'. By a *zipper* we mean the square of a path P of length at least 2; the *square* P^2 of a graph P has the same vertex set as P where two vertices are adjacent exactly when they are at distance 1 or 2 in P.

Lemma 2. *Let $G = (V, E)$ be a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph. For a halfspace A of G and its complement $A' = V - A$, the borders L and L' of A and A', respectively, constitute separating lines such that L and L' together induce a zipper.*

Proof. We first claim that any two adjacent vertices u and v of L have a common neighbor in L'. Indeed, if some neighbors $u', v' \in L'$ of u and v, respectively, are adjacent, then one of them is a common neighbor of u and v because G is bridged; else, if $d(u', v') \geq 2$, then u' and v' have a common neighbor $z \in A'$ adjacent to both u and v because A' is convex and G is bridged.

Since neighborhoods of convex sets are convex in a bridged graph, we infer that the borders $L' = N(A) \cap A'$ and L are convex. For $u, v, w \in L$ with $u, w \in N(v)$, there exist $x, y \in L'$ with $x \in N(\{u, v\})$ and $y \in N(\{v, w\})$, by what has been shown above. Necessarily, u and w are non-adjacent, whereas x and y are distinct adjacent vertices, since L and L' are convex and both C_4 and K_4 are forbidden subgraphs. If v had a third neighbor in L, then a vertex $z \in N(\{t, v\}) \cap A'$ together with x, y, and v would form a forbidden K_4. This shows that L induces a convex path. L evidently separates $A - L$ from A' unless $L = A$. In the latter case, L is a boundary line because the neighborhood of every vertex of L must be a path.

Let P denote the bipartite subgraph of G comprising the edges between L and L' (that is, each having one end vertex in L and the other in L'). By what has been shown, $L \cup L'$ is the vertex set of P, every vertex of P has degree 2 except for two vertices (that are end vertices of L or L') which have degree 1, and P is connected. Since G is K_4-free and L and L' induce convex paths, we conclude that P is a path, whence $L \cup L'$ induces a zipper given by P^2. $\square$

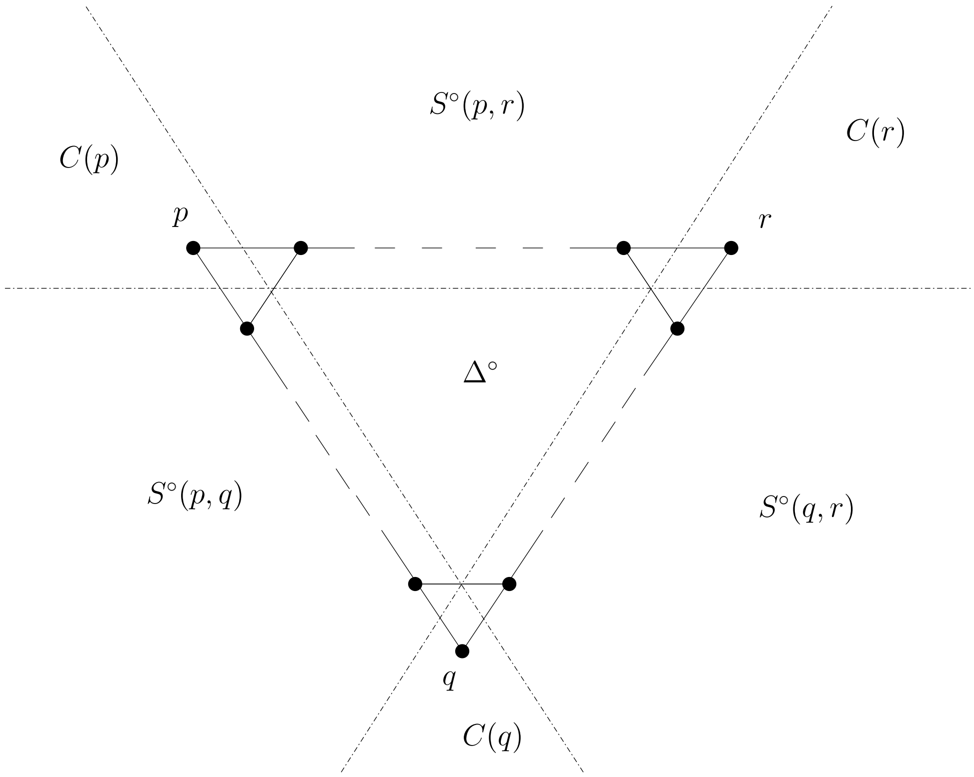


FIGURE 3. Convex partition with respect to a metric triangle pqr

Proposition 4. *Let pqr be a metric triangle of size $k \geq 1$ in a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph $G = (V, E)$, which gives rise to a k-deltoid $\Delta = \operatorname{conv}(p, q, r)$.*

(a) *The sides $I(p,q), I(p,r)$, and $I(q,r)$ of Δ extend to border lines $L(p,q), L(p,r)$, and $L(q,r)$ of halfspaces $H(p,q), H(p,r)$, and $H(q,r)$, respectively, such that*
$$\Delta \cap H(p,q) = I(p,q),\ \Delta \cap H(p,r) = I(p,r),\ \Delta \cap H(q,r) = I(q,r).$$

The three border lines pairwise intersect only in one corner of Δ *each and induce a partition of the vertex set* V *into the following seven convex sets: the* $(k-2)$*-deltoid* Δ° *(for* $k \geq 2$*), the (nonempty) cones*

$$C(p) = H(p,q) \cap H(p,r),\ C(q) = H(p,q) \cap H(q,r),\ C(r) = H(p,r) \cap H(q,r),$$

and the open sectors

$$S^\circ(p,q) = H(p,q) - (C(p) \cup C(q)),$$
$$S^\circ(p,r) = H(p,r) - (C(p) \cup C(r)),$$
$$S^\circ(q,r) = H(q,r) - (C(q) \cup C(r));$$

see Fig. 3. *Moreover, the union of any open sector with one of its two neighboring cones is convex. Each of the corresponding closed sectors*

$$S(p,q) = H(p,q) - [(C(p) - L(p,r)) \cup (C(q) - L(q,r))],$$
$$S(p,r) = H(p,r) - [(C(p) - L(p,q)) \cup (C(r) - L(q,r))],$$
$$S(q,r) = H(q,r) - [(C(q) - L(p,q)) \cup (C(r) - L(p,r))]$$

is also convex as well as its union with Δ *or with any neighboring cone.*

(b) *The three border lines pairwise recombined at the corners of* Δ *yield altogether six new shortest paths, such as* $(L(p,r) \cap C(p)) \cup (L(p,q) - C(p))$*. Moreover, the following statements hold for every vertex* $u \in C(p)$:

$$C(p) = p/q \cap p/r,$$
$$\Delta \cup S(q,r) \subseteq p/u,$$
$$S(q,r) \subseteq I(q,r)/u \subseteq I(q,r)/p = H(q,r),$$
$$S^\circ(q,r) = I^\circ(q,r)/u \text{ if } k \geq 2.$$

(c) *Any two lines* L_1 *and* L_2 *of* G *that pass through* $I(p,q) - \{p\}$ *and* $I(p,r) - \{p\}$ *are disjoint whenever* $L_1 \cap \Delta$ *and* $L_2 \cap \Delta$ *are different.*

Proof. (a): Let q' and r' be the (adjacent) neighbors of p on the paths $I(p,q)$ and $I(p,r)$, respectively. Then, by virtue of [3, Lemma 12], there are unique halfspaces $H(p,q'), H(p,r')$, and $H(q',r')$ that intersect the triangle $\{p,q',r'\}$ exactly in $\{p,q'\}, \{p,r'\}$, and $\{q',r'\}$, respectively. $H(p,q')$ contains q but not the neighbor q'' of q in $I(q,r)$ because $r' \in I(p,q'')$. Therefore $H(p,q')$ is the unique halfspace $H(p,q)$ that includes $I(p,q)$ and is disjoint from the $(k-1)$-deltoid $\Delta - I(p,q)$. Analogously, $H(p,r')$ is the required halfspace $H(p,r)$. The assertion for $H(q,r)$ is settled by symmetry. If $H(q,r)$ and $V - H(q',r')$ intersect in a vertex x, then x is closer to p than to q' and closer to q than to the neighbor of q in $I(p,q)$. Hence the path $I(p,q)$ must include a subpath

s_1, s_2, s_3 with $2d(x, s_2) > d(x, s_1) + d(x, s_3)$. This, however, conflicts with $I(p, q)$ being convex because G is bridged and weakly modular. Therefore $H(q, r) \subseteq H(q', r')$ holds.

The border lines $L(p, q), L(p, r)$, and $L(q, r)$ of $H(p, q), H(p, r)$, and $H(q, r)$, respectively, intersect Δ in the corresponding sides of Δ. The lines $L(p, r)$ and $L(q, r)$ meet along a convex path within the cone $C(r)$. Suppose that $L(p, r) \cap L(q, r) \neq \{r\}$. Then this intersection contains some neighbor w of r in $C(r)$. Let $s \in I(p, r)$ and $t \in I(q, r)$ denote the two neighbors of r in Δ. Since $L(q, r)$ is the border of $H(q, r)$, the vertices r and w have a common neighbor v in $V - H(q, r)$ by Lemma 2. Moreover, s and v must be adjacent vertices on the border of $V - H(q, r)$. Then, however, $v \in I(s, w) \subseteq L(p, r)$ contradicts the fact that $L(p, r)$ is a line.

If the intersection of $H(p, q'), H(p, r')$, and $H(q', r')$ had some vertex z in common, then z must be at equal distance to p, q', and r'. Then, as G is weakly modular and C_4-free, we would obtain a common neighbor of p, q', and r' (one step closer to z), thus yielding a forbidden K_4. We conclude that, in particular,

$$H(p, q) \cap H(p, r) \cap H(q, r) = \emptyset,$$

whence the three cones and open sectors are pairwise disjoint. Each open sector is convex because it is the intersection of three halfspaces; for instance,

$$S^\circ(p, q) = H(p, q) - ((H(p, r) \cup H(q, r)).$$

Then, for instance, $S^\circ(p, q) \cup C(q) = H(p, q) - H(p, r)$ is convex as well. In order to show that the cones and open sectors together with Δ° form a partition of V, it remains to verify that

$$\Delta^\circ = V - [H(p, q) \cup H(p, r) \cup H(q, r)].$$

Clearly Δ° is disjoint from the halfspaces $H(p, q), H(p, r)$, and $H(q, r)$. On the other hand, it follows from [3, p. 708] that any vertex $x \notin \Delta$ is located outside the region bounded by the sides of Δ in the canonical representation of G in the plane. Therefore, if $\Delta^\circ \neq \emptyset$, any shortest path connecting x with a vertex of Δ° intersects one of the sides of Δ, say $I(p, q)$, whence x does not lie in the halfspace $V - H(p, q)$. Finally, if $\Delta^\circ = \emptyset$, then the vertices p, q, and r are pairwise adjacent, and x cannot be equidistant from the corners of Δ, say $d(x, p) < d(x, r)$, whence x belongs to $H(p, q)$. This proves the desired equality for Δ°.

From the definition of $S(p, q)$ and $S^\circ(p, q)$ we infer that

$$S(p, q) = S^\circ(p, q) \cup (H(p, q) \cap L(p, r)) \cup (H(p, q) \cap L(q, r)).$$

Taking the union with $\Delta = (\Delta^\circ \cup I^\circ(p,q)) \cup (I(p,r) - \{r\}) \cup (I(q,r) - \{r\}) \cup \{r\}$ this yields

$$\begin{aligned} S(p,q) \cup \Delta &= (S^\circ(p,q) \cup \Delta^\circ) \cup (L(p,r) - H(q,r)) \cup (L(q,r) - H(p,r)) \cup \{r\} \\ &= [(V - H(p,r)) \cap (V - H(q,r))] \cup [L(p,r) - H(q,r)] \\ &\quad \cup [(L(q,r) - H(p,r)] \cup [L(p,r) \cap L(q,r)] \\ &= [(V - H(p,r)) \cup L(p,r)] \cap [(V - H(q,r)) \cup L(q,r)]. \end{aligned}$$

Therefore $S(p,q) \cup \Delta$ is convex because the border of any halfspace together with the complementary halfspace constitute a convex set. Moreover, as $S(p,q) = (S(p,q) \cup \Delta) \cap H(p,q)$, we conclude that $S(p,q)$ is convex. $S(p,q) \cup C(q)$, for instance, is also convex because it is the intersection of $H(p,q)$ and the neighborhood of $V - H(p,r)$. This completes the proof of (a).

(b): Suppose by way of contradiction that $(L(p,r) \cap C(p)) \cup (L(p,q) - C(p))$ is not a shortest path. Then we can select two non-adjacent vertices x and y such that the interval $I(x,y)$ intersects this path exactly in x and y. Since $L(p,r) \cap C(p)$ and $L(p,q) - C(p) = L(p,q) \cap (V - H(p,r))$ are convex paths by part (a), we can assume that $x \in L(p,r) \cap C(p) - \{p\}$ and $y \in L(p,q) - C(p)$. Then, as $I(p,x) \subseteq L(p,r)$ and $I(p,y) \subseteq L(p,q)$, the vertices x, y, and p form a metric triangle. Consequently, the neighbors of p in $I(p,x)$ and $I(p,y)$ are adjacent. Since the neighbors of p in $I(p,r)$ and $I(p,y)$ are also adjacent, this contradicts the convexity of $L(p,r)$, thus establishing the first assertion in (b).

If $u \in p/q \cap p/r$, that is, $p \in I(q,u) \cap I(r,u)$, then $u \in H(p,r) \cap H(p,q) = C(p)$, because $q \in V - H(p,r)$ and $r \in V - H(p,q)$. Conversely, let $u \in C(p)$. Any shortest path P from u to r intersects either the path $L(p,r) \cap C(p)$ or the path $L(p,q) \cap C(p)$ in some vertex x. Then $p \in I(x,r)$ because the subpath from x to r on $L(p,r)$ or $(L(p,q) \cap C(p)) \cup (L(p,r) - C(p))$, respectively, is a shortest path. Therefore, in either case $u \in p/r$. Analogously, we obtain $u \in p/q$.

For $u \in C(p) = p/q \cap p/r$, we immediately get $p, q, r \in p/u$ and hence $\Delta \subseteq p/u$ by convexity of shadows. Then it follows that $I(q,r)/u$ is contained in both p/u and $I(q,r)/p$. The halfspace $H(q,r)$ necessarily includes $I(q,r)/p$. As to the converse, consider a shortest path P from $z \in H(q,r)$ to p. If P intersects $I^\circ(q,r)$, then certainly z belongs to $I(q,r)/p$. Otherwise, P meets either $L(q,r) \cap H(p,r)$ or $L(q,r) \cap H(p,q)$, say, the former. Since $I(p,r) \cup (L(q,r) \cap H(p,r))$ is a shortest path that joins p with a vertex from P and passes through r, we infer that $z \in I(q,r)/p$. A similar argument, applied to $z \in S(q,r)$ and $u \in C(p)$, shows that $z \in I(q,r)/u$ because $L(p,q) \cap H(q,r)$ and $L(p,r) \cap H(q,r)$ are contained in p/u, whence $S(q,r) \subseteq I(q,r)/u$.

To prove the equality claimed for $S^\circ(q,r)$, let q' and r' be the neighbors of q and r in $I(q,r)$. Then, as $I^\circ(q,r)/u$ does not contain q and r, we infer from what has been shown that $I^\circ(q,r)/u$ is included in $H(q,r) - [C(q) \cup C(r)] = S^\circ(q,r)$. To show the converse, recall that $S^\circ(q,r) \subseteq I(q,r)/u$. Assume that some shortest path connects u with a vertex $x \in S^\circ(q,r)$ that passes through q, say. Since $q \in H(p,q)$ but $x \in V - H(p,q)$, the interval $I(q,x)$ contains either q' or a common neighbor y of q and q' on the border of $V - H(p,q)$, by Lemma 2. In either case we get $q' \in I(u,x)$, as required.

(c): Any line L of G that contains q and some vertex of $I(p,r) - \{p\}$ necessarily intersects Δ in the side $I(q,r)$. Then L stays within $H(q,r)$ because it cannot intersect $C(p)$, or $S^\circ(p,q)$, or $S^\circ(p,r)$, by the first equality of part (b) and last statement of (b), respectively. A line L' of G passing through the open sides $I^\circ(p,q)$ and $I^\circ(p,r)$ of Δ cannot contain any of the three corners of Δ because it is a convex path. Hence L' is disjoint from the three cones, by the first inclusion in statement (b). Hence, if L' contained a vertex from $S^\circ(q,r)$, then L' would pass through $I^\circ(q,r)$, which is impossible. Therefore $L' \subseteq V - H(q,r)$, namely, L' is included in the union of $S^\circ(p,q), S^\circ(p,r)$, and $\Delta - I(q,r)$. Clearly, $L' \cap \Delta$ is a line of Δ that is uniquely determined by its end point in $I^\circ(p,q)$ as well as by its other end point (in $I^\circ(p,r)$). Assume that $x_i \in I^\circ(p,q)$ and $y_i \in I^\circ(p,r)$ are the end points of $L_i \cap \Delta$ ($i = 1,2$). If $L_1 \cap \Delta$ and $L_2 \cap \Delta$ are different, then they constitute the sides of two distinct sub-deltoids $\Delta_1 = \mathrm{conv}(p, x_1, y_1)$ and $\Delta_2 = \mathrm{conv}(p, x_2, y_2)$ of Δ, where, say, $\Delta_1 \subset \Delta_2$. When we apply the preceding observations to Δ_2 instead of Δ and let L_1 and L_2 play the roles of L and L', we can conclude that $L_1 \subseteq V - H(x_2, y_2)$ and $L_2 \subseteq H(x_2, y_2)$, so that L_1 and L_2 do not meet. $\square$

Lemma 3. *Let pqr be a metric triangle of size $k \geq 1$ in a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph $G = (V, E)$.*

(a) *If $v \in C(q)$ and $w \in C(r)$, then $(pvw) = p$. In particular, if $u \in C(p)$ and $p \in I(u,v) \cap I(u,w)$, then $(uvw) = p$.*
(b) *If $v \in C(q)$ and $y \in S(p,r)$, then $I(v,y) \cap I(p,r) = I((pry),(rpy))$ and $(vry) = (qr(rpy))$.*
(c) *If $u \in C(p)$ and $v, w \in C(r)$, then $(uvw) \in S^\circ(p,r) \cup C(r)$.*
(d) *If y is a vertex of $H(p,r)$, then $(ypq) = (rp(qpy))$.*

Proof. We denote the convex hull of $\{p,q,r\}$ by Δ.
(a): The first statement follows from

$$I(p,v) \cap I(p,w) \subseteq (S(p,q) \cup C(q)) \cap (S(p,r) \cup C(r)) = \{p\},$$

by Proposition 4(a).

(b): From the inclusion $S(p,r) \subseteq I(p,r)/v$ established in Proposition 4(b) we obtain $I(v,y) \cap I(p,r) \neq \emptyset$. Since $I(p,r) \subseteq q/v$, all vertices of $I(p,r)$ have the same distance to v. On the other hand, letting $p' = (pry)$ and $r' = (rpy)$, the vertex y is equidistant to all vertices of the path $I(p',r')$ and has a larger distance to the remaining vertices of $I(p,r)$. Therefore $I(v,y) \cap I(p,r) = I(p',r')$, as required. Note that, as a consequence, $I(q,y) \cap \Delta = \mathrm{conv}(q,p',r')$.

By what has just been shown, there exists a shortest path from v to y via q, $q' = (qrr')$, and r'. Therefore q' belongs to $I(v,r) \cap I(v,y)$. Then, as

$$\begin{aligned} I(q',r) \cap I(q',y) &\subseteq I(q',r) \cap \Delta \cap I(q,y) \\ &= I(q',r) \cap \mathrm{conv}(q,p',r') = \{q'\}, \end{aligned}$$

we obtain $q' = (vry)$.

(c): Suppose the assertion is false. Then without loss of generality we may assume that uvw is a metric triangle (of size ≥ 1). The sides $I(u,v)$ and $I(u,w)$ of the deltoid $\mathrm{conv}(u,v,w)$ intersect the line $L(q,r) \cap H(p,r)$ of $H(p,r)$ in two distinct vertices y and z such that, say, $y \in I(r,z)$. Further, $I(u,w)$ and $I(u,v)$ meet the line $L(p,q) \cap H(p,q)$ of $H(p,q)$ in unique vertices x and x', respectively. Then $\mathrm{conv}(x,q,z)$ is a deltoid, which intersects the convex paths $I(u,w)$ and $I(u,v)$ in the different paths $I(x,z)$ and $I(x',y)$. From Proposition 4(c) we infer that any two lines of G extending $I(u,w)$ and $I(u,v)$, respectively, must be disjoint, which is absurd.

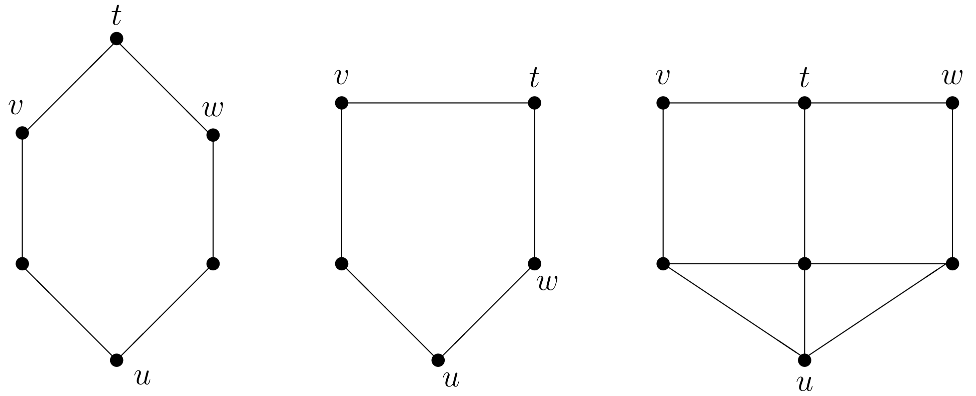
(d): If $y \in C(p)$, then $(rp(qpy)) = (rpp) = p = (ypq)$, as required. If $y \in C(r)$, then $(ypq) = r = (rpq) = (rp(qpy))$ by part (a). Finally, if y belongs to the open sector $S^\circ(p,r)$, then, by the first assertion of (b), (pry) is the vertex of the intersection $I(q,y) \cap I(p,r)$ closest to p, whence $(ypq) = (pry)$. On the other hand, $(qpy) = (qp(pry))$ by the second assertion of (b), and therefore $(rp(qpy)) = (rp(qp(pry)) = (pry)$, completing the proof. ☐

Corollary 1. *Let pqr be a triangle in a finite two-connected, K_4- and sun-free weakly median bridged graph G. If $x \in C(p)$ and $y \in C(q)$, then $I(x,y)$ contains both p and q.*

Proof. $I(r,x) \cap I(r,y) = \{r\}$ by Lemma 3(a). Since $r \notin I(x,y)$ and G is sun-free, the triplet r,x,y has a (unique) quasi-median of size 1. Then, as $p \in I(r,x)$ and $q \in I(r,y)$, this quasi-median is the triangle rpq, as required. ☐

4. Equations in four variables

In order to characterize weakly median graphs among apiculate graphs algebraically, we have to translate the interval conditions defining weak modularity into equations in terms of the imprint operation. Two series of equations then come into play, the first of which (see Lemma 6 below) is implied by axiom 4a of Isbell [12]. Since we focus on equations in at most four variables, we will first have a look at all 4-element algebras realized within apiculate graphs. Note that up to isomorphism there are only two different 3-element subalgebras of the imprint algebras of graphs, viz. the imprint algebras of the path P_2 of length 2 and the triangle K_3.



(a) R_1 (b) R_2 (c) R_3

FIGURE 4. 4-element subalgebras $\{t, u, v, w\}$ of imprint algebras

Lemma 4. *The following list describes (up to isomorphism) all* 4*-element subalgebras* $R = \{t, u, v, w\}$ *of the imprint algebras of apiculate graphs, relative to the number* n *of triangle subalgebras of* R*:*

($n = 0$) *the imprint algebras of the path* P_3*, the star* $K_{1,3}$*, and the cycle* C_4*;*

($n = 1$) *the imprint algebra of the triangle with an edge attached, and the subalgebra* R_1 *(Fig.* 4(a)*) of the* C_6 *algebra;*

($n = 2$) *the* $K_{1,1,2}$ *algebra, and the subalgebra* R_2 *(Fig.* 4(b)*) of the* C_5 *algebra;*

($n = 3$) *the subalgebra* R_3 *(Fig.* 4(c)*) of the imprint algebra of the amalgam of two house algebras along a convex* 2*-path;*

($n = 4$) *the* K_4 *algebra.*

Proof. If the algebra $R = \{t, u, v, w\}$ can be realized as a graph, that is, R is the imprint algebra of a 4-vertex graph, then we have one of the six graphs described in the lemma. Henceforth let $R_n = \{t, u, v, w\}$ be a (proper) subalgebra of the imprint algebra of some apiculate graph G_n such that R_n is different from the preceding six algebras and R_n harbors exactly n distinct triangle subalgebras (which thus constitute metric triangles in G_n). Then R_n must be a *quasi-trivial algebra*, that is, $(xyz) \in \{x, y, z\}$ for all $x, y, z \in R_n$. Necessarily, $1 \leq n \leq 3$ holds, and we may assume that uvw is a metric triangle in G_n. Hence t together with any two from u, v, w forms either the K_3 algebra or the P_2 algebra.

Case $n = 1$: Then, as $\{t, u, v, w\}$ is not a P_3 algebra, at least one of the three P_2 algebras $\{t, u, v\}, \{t, u, w\}$, and $\{t, v, w\}$ has t in between the other two vertices, say $t \in I(v, w)$. Consequently, v and w must belong to $I(t, u)$, whence R_1 can be realized within the 6-cycle G_1 as indicated in Fig. 4(a).

Case $n = 2$: Let $\{t, u, v\}$ be the second triangle subalgebra of R_2. If $I(t, w)$ contains v, then it also includes u, whence R_2 would be the imprint algebra of K_4 minus an edge, contrary to the hypothesis. Therefore neither u nor v belongs to $I(t, w)$, whence $t \in I(v, w)$ or $w \in I(v, t)$, say, the former holds. Then $w \in I(t, u)$ follows, showing that R_2 can be realized within the 5-cycle algebra G_2 of Fig. 4(b) as claimed.

Case $n = 3$: We can assume that $\{t, u, v\}$ and $\{t, u, w\}$ are triangle subalgebras and that $\{t, v, w\}$ is a path algebra with $t \in I(v, w)$. Then, evidently, R_3 can be represented as shown in Fig. 4(c). $\square$

The algebras R_1, R_2, and the partial algebra R_2^* defined next emerge when the triangle and quadrangle conditions are violated. For instance, the imprint algebras of all cycles of length at least 7 harbor both R_1 and R_2 subalgebras. The partial algebra R_2^* equals R_2 (Fig. 4(b)) except that (vuw) and (vwu) are not specified and thus left undetermined. Observe that R_2 does not occur in a house algebra, but R_2^* is shared by the C_5 and house algebras.

Lemma 5. *Let G be a graph.*

(a) *If G violates* (Q)*, then every intrinsic algebra of G contains a R_1 subalgebra.*
(b) *If G violates* (Q!)*, then the imprint algebra of G contains a R_1 subalgebra.*
(c) *If G violates* (T)*, then every intrinsic algebra of G contains a R_2^* partial algebra. Moreover, when G is apiculate, this partial algebra extends to either a R_2 subalgebra or a house subalgebra of the imprint algebra of G.*

Proof. Consider an instance u, v, w, z that violates condition (Q). We can assume $I(u, v) \cap I(u, w) = \{u\}$, so that uvw is a metric triangle. Then $t = z$ satisfies the additional three betweenness properties required for R_1. Therefore $\{t, u, v, w\}$ constitutes the algebra R_1 with respect to any intrinsic operation of G. If, instead, (Q) is fulfilled for this quartet but with more than one possible choice for x, then there are at least two v-apices relative to u and w, so that the imprint of u and w with respect to v equals v and, analogously, the imprint of u and v with respect to w equals w. We may assume that $(uvw) = u$, so that R_1 arises.

Now consider a triplet u, v, w as described in the triangle condition (T). Then we can assume that uvw is a metric triangle. If this triplet does not admit the desired vertex x, then any neighbor t of w in $I(u, w)$ is at distance 2 to v. Hence $\{t, u, v, w\}$ with (vtu) and (vut) unspecified constitutes a copy of the partial algebra R_2^*, where the roles of t and w are interchanged with respect to Fig. 4(b). $\square$

Lemma 6. *Let G be a graph.*

(a) *If some intrinsic algebra of G satisfies one of the equations*

(A8) $((uwx)(vwx)u) = (uwx)$,

(A8′) $(((uwx)(vwx)u)wx) = ((uwx)(vwx)u)$,

then G is weakly modular.

(b) *If some intrinsic algebra of G satisfies the equation*

(A9) $((uwx)((uvx)wx)u) = (uwx)$,

then G satisfies (Q)*; moreover,* (Q!) *holds provided that the intrinsic operation is the imprint operation.*

(c) *If some intrinsic algebra of G satisfies the equation*

(A10) $((((uvw)vx)wx)(vwx)((uvw)vx)) = (((uvw)vx)wx)$,

then G satisfies (T).

Proof. First observe that (A9) and (A10) are particular instances of (A8).

Suppose that R_1 is a subalgebra of some intrinsic algebra of G, where now u, x, v, w play the roles of t, u, v, w in Fig. 4(a). Then

$$(uwx) = w, \ (uvx) = v = (vwx), \text{ and } (wvu) = u,$$

whence (A8′) and (A9) are violated.

Finally suppose that the partial algebra R_2^* is found within some intrinsic algebra of G, where now x, v, w, u play the roles of t, u, v, w in Fig. 4(b). Then

$$(uwx) = x, \ (vwx) = v, \text{ and } (xvu) = u = (uvw),$$

whence (A8′) and (A10) are violated.

Summarizing, this shows that under the hypothesis of (a) R_1 and R_2^* are forbidden, whereas for (b) R_1 alone and for (c) R_2^* alone are forbidden. Hence Lemma 5 completes the proof. $\square$

One can derive (A8′) from (A8) by means of (A4): denote the two sides of (A8) by y (left) and z (right), respectively, then $(ywx) = (zwx) = z = y$. Recall that imprint and apex algebras always satisfy (A4). Obviously, when a graph G has diameter 2 (such as the graphs of [2, Fig. 1]), (A4) is fulfilled by all intrinsic operations of G.

In order to verify (A8) for an intrinsic algebra of a given graph, it suffices to check only those quartets u, v, w, x of distinct vertices for which $(uwx) \neq u$ and $(vwx) \neq w, x$. In the case of (A9) we can additionally assume that

$$v = (uvx) \notin I(u, (uwx)) \cup I((uwx), x),$$

provided that the intrinsic operation is an apex operation. Hence, in particular, $v = (uvx)$ and (uwx) then do not lie on a common shortest path between u and x. This immediately proves claim (b) in the following example.

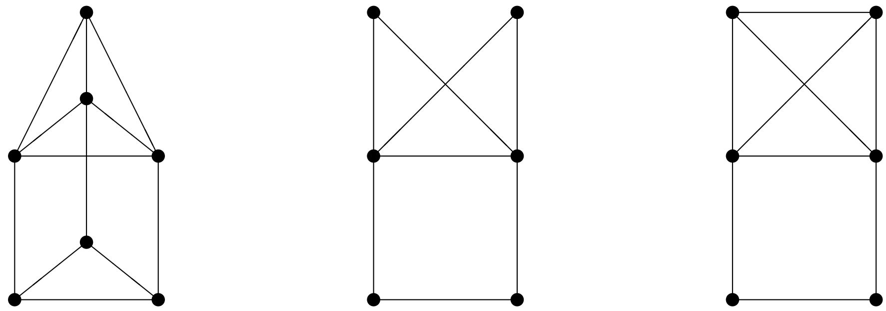
Example 1. (a) (A8) and hence (A8′) are satisfied by the imprint algebras of the graphs of [2, Fig. 1(a,b)] and by all intrinsic algebras of the graphs of [2, Fig. 1(c,d)].
(b) The imprint algebra of a geodesic graph (such as C_5) fulfills (A9).
(c) The C_6 algebra satisfies (A10).

To establish claim (a), first note that the C_4 and $K_{1,1,2}$ subalgebras of the imprint algebras of the graphs of [2, Fig. 1(a,b)] satisfy (A8). Therefore we can assume that the distinct vertices u, v, w, x (where $(uwx) \neq u$ and $(vwx) \neq w, x$) are not covered by either subgraph. $K_{2,3}$ then does not accomodate such a quartet. In the second graph of [2, Fig. 1], the only choices yield $\{v, w, x\}$ as a triangle with u adjacent to exactly one of w and x but non-adjacent to v, so that (A8) is evidently satisfied here.

As for the second assertion in (a), observe that the $K_{1,1,2}$ and K_4 subalgebras satisfy (A8). Hence we can assume (in addition to the above premises) that exactly one of the central vertices of the graph of [2, Fig. 1(c)] or [2, Fig. 1(d)] is from u, v, w, x. If u serves as a central vertex, then we obtain $((uwx)(vwx)u) = ((uwx)vu) = (uwx)$ because (uwx) is either u or a central vertex. In $K_{1,1,3}$, neither u, nor w, nor x could play the role of a central vertex under the premises. But in the other graph $\{v, w, x\}$ could form a triangle with either the vertex w or x being central, which then equals both sides of (A8).

As for (b), the vertices (uwx) and (uvx) lie on the unique shortest path between u and x. If $(uwx) \in I((uvx), x)$, then also $(uwx) \in I((uvx), w)$ holds, yielding $((uvx)wx) = (uwx)$ because geodesic graphs are apiculate. Consequently, the left-hand side of (A9) is also (uwx). Finally, if $(uvx) \in I((uwx), x)$, then $((uvx)wx) \in I((uwx), x)$ from which we infer that (uwx) is between $((uvx)wx)$ and u, whence the left-hand side of (A9) is again (uwx).

As to (c), if $(uvw) \in \{v, w\}$ or $(vwx) \in \{w, x\}$, then (A10) clearly holds. Therefore we can assume $(uvw) = u$ and $(vwx) = v$ in order to verify (A10) in C_6. Then only the case $(uvx) = u$ remains to be checked for (A10), but this cannot be reconciled with $(vwx) = v$ in the 6-cycle.



(a) (b) (c)

FIGURE 5. Imprint algebras enjoying (A9) and (A11)

The preceding example shows that (A8) does not imply (A5), and that (T) is not a consequence of (A9) plus (Q), and (Q) not a consequence of (A10) plus (T).

Proposition 5. *The imprint algebra of an apiculate graph G of diameter 2 satisfies* (A9) *if and only if G does not contain an induced subgraph isomorphic to the graph of [2, Fig.* 4(b)*] or its companion that has the additional chord tu. In particular, the imprint algebras of the house and the graphs of Fig.* 5 *satisfy* (A9)*.*

Proof. The two forbidden graphs [2, Fig. 4(b)] where tu is now a potential chord are apiculate and of diameter 2 but evidently their imprint algebras violate (A9): the left-hand side becomes u whereas the right-hand side is y.

As to the converse, let the quartet u, v, w, x violates (A9). Then, by the diameter constraint, we can assume that $d(u, x) = 2$, and, moreover, $v = (vwx)$ and $y = (uwx)$ are two distinct common neighbors of u and x. Since (yvu) is then the left-hand side of (A9) and y equals the right-hand side, these two vertices are different by hypothesis, and therefore u, v, y, x induce a 4-cycle in G. The vertex w must be a neighbor of y but cannot be adjacent to u, x, or v (because the graphs of [2, Fig. 1(a,b)] are forbidden). Hence there exists a common neighbor t of v and w in G. In order to avoid an induced subgraph from [2, Fig. 1], the vertex t is nonadjacent to y and adjacent to at most one of u, x. □

The second suite of equations that we will later employ in the algebraic characterization rejects R_2 and R_3 subalgebras.

Lemma 7. *If some intrinsic algebra of a graph G fulfills any one of the three equations*

(A11) $((wux)(uvw)v) = (w(uvw)(vu(wux)))$,

(A11$'$) $((wux)(uvw)v) = (w(uvw)(v(uvw)(wux)))$,

(A11$''$) $((wux)(uvw)(vuw)) = ((wuv)(uvw)((vuw)(uvw)(wux)))$,

then it cannot contain a R_2 or a R_3 subalgebra. If some apex algebra of G fulfills (A11) *or* (A11$'$)*, then G is apiculate.*

Proof. Suppose that some intrinsic algebra of G includes a R_2 subalgebra $\{u, v, w, x\}$ such that w, u, v, x play the roles of t, u, v, w in Fig. 4(b). Then the left-hand sides of (A11) and its two variants all equal $(xuv) = x$, whereas the right-hand sides are equal to $(wuv) = w$. Hence all three equations are violated.

To see that G is apiculate whenever (A11) or (A11$'$) holds for some apex operation, consider any vertex $x \in I(u, v) \cap I(u, w)$ such that $I(x, v) \cap I(x, w) = \{x\}$. We wish to show that x and (uvw) coincide. We may assume that $I(v, x) \cap I(v, (uvw)) = \{v\}$. Denote the left-hand side of (A11) and (A11$'$) by $p = p'$, and the corresponding right-hand sides by q and q'. Then

$$\begin{aligned} p = p' &= (x(uvw)v), \\ q &= (w(uvw)x), \\ q' &= (w(uvw)v). \end{aligned}$$

If (A11) holds, then $p = q \in I(v, x) \cap I(w, x) = \{x\}$, whence $x \in I((uvw), w)$ and consequently, $x = (uvw)$ since (uvw) is a u-apex relative to v and w. If instead (A11$'$) holds, then

$$p = q' \in I(v, x) \cap I((uvw), w) \subseteq I(u, v) \cap I((uvw), w) = \{(uvw)\},$$

whence $(uvw) \in I(v, x)$ and consequently, $x = (uvw)$. This shows that under either hypothesis the graph G is apiculate. □

Proposition 6. *For Pasch graphs the equations* (A11),(A11$'$), *and* (A11$''$) *are all equivalent.*

Proof. From [2, Proposition 2] we know that a Pasch graph is apiculate. We may assume that $x = (wux)$. Put $t = (vux)$, $t' = (v(uvw)x)$, and $t'' = ((vuw)(uvw)x)$, so that the sides of the three equations become

$$\begin{aligned} p = p' &= (x(uvw)v), \\ p'' &= (x(uvw)(vuw)), \\ q &= (w(uvw)t), \\ q' &= (w(uvw)t'), \\ q'' &= ((wuv)(uvw)t''). \end{aligned}$$

The shadow $(vuw)/v$ trivially contains u, w, and (vuw). Since point-shadows are convex, we have $(uvw), (wuv), x \in I(u, w) \subseteq (vuw)/v$. Therefore $(vuw)/v$ includes $p = p', p'', q, q', q'', t''$, as well as t, t' (because the graph is apiculate), whence $p = p'' \in I(x, (vuw))$ and $t' = t'' \in I((vuw), (uvw))$. Then $t, (uvw) \in I(u, t')$. Further, $t, (vuw), (uvw) \in I(u, v) \subseteq (wuv)/w$. Because this point-shadow then contains t' as well, we infer that it also harbors q and q', whence $q, q' = q'' \in I((uvw), (wuv))$. This, en passant, establishes the equivalence of (A11$'$) and (A11$''$). Now, since $t \in I(u, t')$ and $u, t' \in q'/w$, the point-shadow q'/w contains t. Hence $q' \in I(w, t) \cap I(w, (uvw))$, and as a consequence

$$q \in I(q', (uvw)) \cap I(q', t).$$

The (convex) point-shadow p/x contains the point t' because $t' \in I((uvw), v) \subseteq p/x$. Since $t' \in I(v, t) \subseteq I(v, x)$, we can summarize this information by

$$p, t \in I(t', x) \subseteq I(v, x).$$

Now assume that $p = q$ holds. Because $q \in I((uvw), (wuv)) \subseteq I((uvw), w)$, we have $(uvw) \in I(u, p) = I(u, q)$. Then, as $t \in I(u, t')$, the intervals $I(p, t)$ and $I(t', (uvw))$ have a vertex z in common by the Pasch axiom applied to $u, t', p, t, (uvw)$. Then, as p and t belong to the (convex) interval $I(t', x)$, we infer $z \in I(t', x) \cap I(t', (uvw))$. Since t' is the v-apex relative to x and (uvw), we conclude that $z = t'$. Hence $t' \in I(t, p) = I(t, q) \subseteq I(t, w)$, and consequently, $p \in I(t', w)$. Therefore $p = q \in I(w, t') \cap I(w, (uvw)) = I(w, q')$. On the other hand, we know that $p = q \in I(q', (uvw))$. This entails $q' = q$.

Finally assume that $p = q'$ holds. Then $q \in I(q', t) = I(p, t) \subseteq I(t', x) \subseteq I(x, v)$ by convexity of intervals. Since $q \in I(q', (uvw))$ and $q' = p = p'' \in I(x, (uvw))$, we infer $q \in I(x, (uvw)) \cap I(x, v) = I(x, p) = I(x, q')$ by convexity of $I(x, (uvw))$, whence $q' \in I(q, (uvw))$, so that $q = q'$ follows. □

From the proof we infer that (A11$'$) and (A11$''$) are equivalent for apiculate graphs having convex point shadows. Pasch graphs in general need not satisfy any of the equations (A8),(A9),(A11),(A11$'$), and (A11$''$) as the 5-cycle C_5 shows. In order to verify (A11$''$) for some Pasch graph (or more generally, a graph with unique quasi-medians), one may assume that

$$v = (vuw) \neq (uvw) \text{ and } x = (wux) \notin (uvw)/v \cup I(w, (wuv))$$

because $(uvw)(vuw)(wuv)$ is then the quasi-median of u, v, w. A quartet u, v, w, x of vertices with these properties does not exist in C_6, for instance. For the house, only one quartet (up to automorphism) is feasible, viz. u, v, w, and $x = z$ as labelled in [2, Fig. 4(a)]; in this case, $(xuv) = x = (wut) = (wu(vux))$, as required for (A11$''$). All feasible quartets in the graphs of Fig. 5 are already included in some convex prism or house. We summarize and extend these observations in the following example.

Example 2. (a) The imprint algebras of the graphs of [2, Fig. 1] satisfy (A11) and its variants. An apex algebra of any of these graphs satisfies (A11$''$) exactly when it is a priority apex algebra.
(b) The C_6 algebra, the house algebra, and the imprint algebra of the graphs of Fig. 5 satisfy (A11) and its variants.

As for (a), we may assume $(uvw) \neq (wuv)$ and $x = (wux) \neq (uvw)$. Then the first assertion in (a) is quite evident. (A11$''$) can potentially be violated in an apex algebra of $K_{1,1,3}$ only when u, v, w are the three vertices of degree 2. Let y and z denote the two central vertices of $K_{1,1,3}$. Then we may assume that x is either u, w, or y. If the apex operation has the priority property, then either side of (A11$''$) equals the vertex from $\{y, z\}$ that has higher priority. If the priority property does not hold, then we may assume $(vuw) = y$ but $(uvw) = (wuv) = z$, yielding z on the right-hand side of (A11$''$) and x on the other.

Another type of equations describes the key features of metric triangles more directly, as expressed by the billiard law in deltoids. The resulting equations in the following lemma are a bit lengthy but rather easy to handle.

Lemma 8. (a) *If some intrinsic algebra of a graph G satisfies*

(A12) $\quad (uw(wv(vu(uw(wv(vu(uwx))))))) = ((uvw)(wuv)(uwx)),$

or the weaker equation

(A12$'$) $\quad (u'w'(w'v'(v'u'(u'w'(w'v'(v'u'x')))))) = x'$

$$\textit{where } u' = (uvw), v' = (vwu'), w' = (wu'v'), \textit{ and } x' = (u'w'x),$$

then G is weakly modular.
(b) *If some apex algebra of G satisfies* (A12)*, then G is weakly median.*
(c) *If the imprint algebra of G satisfies* (A12$'$)*, then G satisfies* (Q!) *and* (T!).

Proof. (a): Recall that u', v', w' as defined in (A12′) form a quasi-median of u, v, w. Then clearly (A12) implies (A12′).

Suppose that R_1 is a subalgebra of some intrinsic algebra of G, where now x, u, v, w play the roles of t, v, u, w in Fig. 4(a). Then $u' = u, v' = v, w' = w$, and $x' = x$, whence the right-hand side of (A12′) equals x, whereas the left-hand side is u. Therefore (A12′) implies (Q) in view of Lemma 5(a).

Now consider a triplet u, v, w as described in the triangle condition (T) but suppose that uvw is a metric triangle. Let x be a neighbor of w in $I(u, w)$. Again, $u' = u, v' = v, w' = w$, and $x' = x$, whence the right-hand side of (A12′) is x. If $(vux) = v$, then the left-hand side of (A12′) is equal to w. Otherwise $(vux) = y$ is adjacent to v and x but not adjacent to w. Then $(wvy) = v$, and we conclude that the left-hand side of (A12′) equals w, so that (A12′) is violated in either case.

(b): The graph G is weakly modular by what has just been shown. Suppose that G contains some unconnected triplet u, v, w having at least two common neighbors.

Case 1: u, v, and w are pairwise nonadjacent. If $|\{(uvw), (vuw), (wuv)\}| \leq 2$, then, say, $y = (uvw) = (wuv)$ is a common neighbor of u, v, and w. Let x be another common neighbor. Then the left-hand side of (A12) is x and the right-hand side equals $(yyx) = y$, yielding a contradiction. So, $y = (uvw), z = (wuv)$, and $x = (vuw)$ are all different such that, say, $x \notin I(y, z)$. Then the right-hand side (yzx) of (A12) is different from the vertex x, which is the left-hand side, again yielding a contradiction.

Case 2: u and v are adjacent. Then w is non-adjacent to u and v, whence $y = (wuv)$ is some common neighbor of u, v, and w. Pick another common neighbor x. Then $(uw(wv(vu(uwx)))) = (uwv) = u$ and further $(uw(wv(vuu))) = (uwy) = y$, but $((uvw)(wuv)(uwx)) = (uyx) = u$, contradicting (A12).

(c): Since a R_1 subalgebra cannot occur under (A12′), G satisfies (A12′) by Lemma 5(b). If the weakly modular graph G violates (T!), then we obtain one of the graphs of [2, Fig. 1(b,d)] as an induced subgraph. For u, v, w, x as is indicated in that figure we obtain u as the left-hand side of (A12′) but x as the right-hand side. ☐

Example 3. (a) Any apex algebra of a pseudo-modular graph satisfies (A12′).
(b) The imprint algebras of $K_{2,3}$ and $K_{1,1,3}$ satisfy (A12).

Statement (a) is obvious because all metric triangles of a pseudo-modular graph have size at most 1 by definition. As for (b), when u, v, and w induce a path or triangle in $K_{2,3}$ or $K_{1,1,3}$, these vertices together with (uwx) are included in some C_4 or $K_{1,1,2}$ subalgebra, which evidently satisfies (A12). Otherwise, u, v, and w are the vertices of degree 2, so that $(uvw) = u, (vuw) = v$, and $(wuv) = w$. Then both sides of (A12) equal (uwx), no matter whether (uwx) is u, or w, or a common neighbor of u, v, and w.

5. Equational characterization of weakly median graphs

The equations (A8)-(A12) and their variants constitute a sufficiently rich pool from which various characterizations of weakly median graphs (as well as subclasses) can spring.

Theorem 1. *The following statements are equivalent for a graph G :*

(i) *G is weakly median;*
(ii) *some intrinsic algebra of G satisfies* (A5), (A9), *and* (A10);
(iii) *some intrinsic algebra of G satisfies* (A5) *and* (A8)*;*
(iv) *some intrinsic algebra of G satisfies* (A5) *and* (A12)*;*
(v) *some apex algebra of G satisfies* (A12)*;*
(vi) *some apex algebra of G satisfies* (A8) *and* (A11).

In conditions (ii),(iii),(iv), *and* (vi), *the equations* (A5),(A8), (A11), *and* (A12) *may each be substituted by the corresponding variants* (A5′),(A8′), (A11′), *and* (A12′).

Proof. If any variant of one of the conditions (ii)-(vi) is satisfied, then G is weakly modular by Lemmas 6 and 8. If, in addition, (A5) or (A5′) is satisfied, then G is apiculate by [2, Proposition 1] and hence is weakly median. Condition (v) implies that G is weakly median by Lemma 8. Under condition (vi) or its variants, G is apiculate by Lemma 7.

Conversely, we need to show that the imprint algebra of a weakly median graph G satisfies all of the equations listed in the theorem. We already know that (A5) and (A5′) are satisfied. Since (A12) implies (A12′) and (A8) implies (A9) and (A10), it then remains to verify the three equations (A8),(A11″), and (A12), by virtue of Proposition 6. In view of [2, Theorem 1] and Proposition 2, we may assume that G is prime and finite. Four vertices in a complete graph, or a hyperoctahedron, or a 5-wheel either induce a decomposable graph (C_4 or $K_{1,2}$) or a complete graph K_n ($1 \leq n \leq 4$), or are included in a fan (see Fig. 6(a) below).

The case when $G \cong K_n$ is readily checked. As to (A8), we may assume that $w \neq x$, so that $(uwx) = u$ is the resulting vertex on either side of (A8). As to (A11″), if $|\{u, v, w\}| \leq 2$, then (A11″) trivially holds. Else, both sides of (A11″) equal u for $u = x$ and equal w for $u \neq x$. As to (A12), we already know that (A12) holds when u, v, and w form a triangle. If two of u, v, and w are equal, then this vertex is returned by either side of (A12).

Now, assume that G is a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph. To establish (A8), we may stipulate that u, v, w, x are different such that $(uwx) \neq u$

and $(vwx) = v$. Suppose that (A8) is violated: then $((uwx)vu)) \neq (uwx)$. Pick any neighbor u' of (uwx) in $I((uwx), ((uwx)vu)) \subseteq I((uwx), u)$. Then $(u'wx) = (uwx)$ and $((u'wx)vu') = u'$, so that we may substitute u by u'. Recall [3, Lemma 12] that there are exactly two different halfspaces H and H' that contain $(u'wx)$ but not u'. Hence they both include w and x but do not contain v. Since $(wvx), (xvw) \in I(w, x)$, we conclude that $(wvx), (xvw) \in H \cap H'$. Hence the border lines L and L' of H and H' intersect the paths $I(v, (wvx)) - \{v\}$ and $I(v, (xvw)) - \{v\}$ of the deltoid Δ with corners $v, (wvx)$, and (xvw). By Proposition 4(c), L and L' must coincide because they share the vertex $(u'wx)$. The neighbors u' and $(u'wx)$ participate in some triangle with third vertex y because of two-connectivity. Then y is in the symmetric difference of H and H', so that y belongs to one of L and L' but not to the other. This contradicts $L = L'$.

To prove (A11$''$), set $p = (uvw), q = (vuw), r = (wuv)$, and $y = (wux)$. Then Lemma 3(d) applies because $u \in C(p), w \in C(r)$ by Proposition 4(b), and therefore $y = (wux) \in I(u, w) \subseteq H(p, r)$.

Finally, we will establish (A12). Let again pqr denote the quasi-median of u, v, w but now put $y = (uwx)$. Then $y \in I(u, w) \subseteq H(p, r)$. Since $I(u, w)$ is contained in the convex shadows q/v and $I(p, r)/v \subseteq I(p, r)/q$, we conclude that $q \in I(v, y)$ and $\emptyset \neq I(q, y) \cap I(p, r) \subseteq I(v, y) \cap I(p, r)$. We distinguish three cases in regard to the position of y in the halfspace $H(p, r) = C(p) \cup S^\circ(p, r) \cup C(r)$.

Case 1: $y \in C(r)$. The right-hand side of (A12) is the vertex $(pry) = r$. To compute the left-hand side, we first employ Lemma 3(a) to obtain $(vuy) = q$ (because $q \in I(v, u) \cap I(v, y)$). Further, we get $(wvq) = q, (uwq) = p, (vup) = p, (wvp) = r$, and $(uwr) = r$ because pqr is the quasi-median of the triplet u, v, w. Therefore the left-hand side of (A12) equals r as well.

Case 2: $y \in p/v \subseteq p/q$, that is, either $y \in C(p)$ or $y \in S^\circ(p, r) \cap p/v$. Then the right-hand side of (A12) is the vertex $(pry) = p$, which is clear if $y \in C(p)$ and is a consequence of the first statement in Lemma 3(b) otherwise. In the computation of the left-hand side, we first obtain the vertex $z = (vuy) \in I(p, u) \cap I(p, y) \subseteq C(p)$. Since $r \in I(w, v) \cap I(w, z)$, it follows from Lemma 3(a) that $(wvz) = r$. Then, as $(uwr) = r, (vur) = q, (wvq) = q$, and $(uwq) = p$, we eventually see that both sides of (A12) yield the same vertex.

Case 3: $y \in S^\circ(p, r) - p/v$. Then the vertex $p' = (pry)$, which constitutes the right-hand side of (A12), as well as the vertex $q' = (qpp') = (qp(pry))$ are different from p such that $(vpy) = q'$. Then u, v, and y belong to different cones with respect to the deltoid with corners p, q', and p', by virtue of the first equality in Proposition 4(b). Since $q' \in I(v, u) \cap I(v, y)$, we obtain $(vuy) = q'$ by Lemma 3(a) applied to the latter deltoid. Necessarily, $p' \neq r$ and hence $q' \neq q$ because $y \in I^\circ(p,r)/v$ by Proposition 4(b). Note that $x_0 = p'$ and $x_1 = q'$ constitute the first two vertices in the billiard sequence relative to the deltoid Δ with corners $u_0 = p, u_1 = q$, and $u_2 = r$. Moving on, we obtain $x_2 = (rqq')$ and so forth, until we eventually reach $x_6 = x_0 = p'$. In the preceding computation we can actually substitute p, q, r by u, v, w, as we will see next. Since $\Delta \subseteq r/w$ by Proposition 4(b), w belongs to the cone of x_2 with respect to the deltoid with corners x_2, q, and $x_1 = q'$, whence $x_2 \in I(w, x_1)$. Then, as rqp is the quasi-median of w, v, u, we have $x_2 \in I(w,q)$ as well, so that Lemma 3(a) applied to this deltoid yields $(wvx_1) = x_2 = (rqx_1)$. In an analogous fashion we then obtain $(uwx_2) = x_3, (vux_3) = x_4, (wvx_4) = x_5$, and finally $(uwx_5) = x_0 = p'$, so that (A12) is verified. This completes the proof of the theorem. □

None of the conditions (ii)–(vi) can be weakened in a straightforward way. Equation (A5) is indispensable in (ii)–(iv), as can be seen with Examples 1(a) and 3(b). In (ii), both (A9) and (A10) are needed in view of Example 1(c) and Proposition 5. Equation (A12) cannot be replaced by (A12′) in condition (v); see Example 3(a). In (vi), (A8) cannot be weakened to (A9) or (A10) by Proposition 5, Examples 1(c) and 2(b), and (A11) cannot be substituted by (A11″) in view of Example 2(a). From Examples 3(b), 1(a), and 2(a) we deduce that "apex" in (v) or (vi) could not be replaced by "intrinsic".

It is now easy to specify nested subclasses of the class of weakly median graphs by adding stronger equations. Such equations will reject certain prime weakly median graphs as constituents. In some cases, there is a smallest rejected graph that can serve as a forbidden induced subgraph. For example, a weakly median graph for which all constituents are prime pseudo-median graphs is sun-free, i.e., it does not contain the sun (Fig. 1(b)) as an induced subgraph, and vice versa. Recall that the *pseudo-median* graphs are exactly the weakly median graphs in which all quasi-medians have size at most 1 [7]. Now, if one forbids the fan (see Fig. 6(a) below) instead, this excludes the 5-wheel and all two-connected K_4- and $K_{1,1,3}$-free bridged graphs as building stones, so that the prime graphs left are all included in hyperoctahedra. Finally, if the kite (K_4 minus one edge) is forbidden, then the prime constituents are complete graphs, generating all quasi-median graphs.

Proposition 7. *The following statements are equivalent for a graph G:*

(i) *G is weakly median and sun-free;*

(ii) *G is weakly median and its imprint algebra satisfies the equation*

(A13) $\quad ((uvw)(wuv)(uwx)) = ((uvw)(wuv)x);$

(iii) *some apex algebra of G satisfies the equation*

(A14) $\quad (uw(wv(vu(uw(wv(vu(uwx))))))) = ((uvw)(wuv)x).$

Proof. (i) implies (ii): First, let G be a complete graph. If $|\{u, v, w\}| \leq 2$, then (A13) trivially holds. Else, (A13) becomes an instance of (A3′). Now assume that G is a finite K_4- and sun-free weakly median bridged graph. Let $p = (uvw), q = (vuw), r = (wuv)$, and $y = (uwx)$. If $p = q = r$, then both sides of (A13) are equal to this value. So, assume that p, q, and r form a triangle. Note that the vertex x belongs to the open sector $S^\circ(p, q)$ if and only if $d(x, p) = d(x, q) < d(x, r)$; analogous relationships hold for the other two sectors. Therefore, if $x \in C(r) \cup S^\circ(q, r)$, then $r \in I(p, x)$, and the right-hand side of (A13) is $(prx) = r$. If $x \in C(p) \cup S^\circ(p, q)$, then $p \in I(r, x)$, whence $(prx) = p$. Finally, if $x \in C(q) \cup S^\circ(p, r)$, then $d(x, p) = d(x, r)$ and again $(prx) = p$ because p and r are adjacent. The left-hand side of (A13) is the vertex (pry). If $x \in C(r) \cup S^\circ(q, r)$, then $r \in I(u, x) \cap I(u, w)$ by Corollary 1 and Lemma 3(b). Hence $y = (uwx) \in I(r, w) \subseteq C(r)$, yielding $(pry) = r$. If $x \in C(p) \cup S^\circ(p, q)$, then $y = (uwx) \in I(u, x) \cap I(u, w) \subseteq H(p, q) \cap H(p, r) = C(p)$, and therefore $(pry) = p$. If $x \in C(q)$, then $p \in I(u, x) \cap I(u, w)$ by Corollary 1, and therefore $y = (uwx) = p$ by Lemma 3(a), whence $(pry) = p$. Finally, if $x \in S^\circ(p, r)$, then $y \in I(u, w) \subseteq C(p) \cup S^\circ(p, r)$ by Proposition 4(a). If $y \in C(p)$, then $p \in I(r, y)$, so that $(pry) = p$; otherwise, if $y \in S^\circ(p, r)$, then y is equidistant from p and r, yielding $(pry) = p$. We conclude that in each case the two sides of (A13) yield the same vertex.

(ii) implies (iii): Since G is weakly median, equation (A12) is satisfied. Notice that the right-hand side of (A12) equals the left-hand side of (A13), the left-hand sides of (A12) and (A14) coincide as well as the right-hand sides of (A13) and (A14), thus showing that (A14) holds.

(iii) implies (i): The instances of (A12) considered in the proof of Lemma 8 for inferring that G is weakly median all stipulate that $(uwx) = x$, so that (A12) and (A14) coincide in those cases. Therefore G is weakly median and its apex algebra is the imprint algebra. Now, suppose by way of contradiction that G contains a sun with corners u, v, and x. Denote the common neighbor of v and x by w, the common neighbor of u and v by p, and the common neighbor of u and x by y. Then the right-hand side of (A14) is equal to $(pwx) = w$, whereas for the left-hand side we successively compute

$$(uwx) = y,\ (vuy) = p,\ (wvp) = w,\ (uww) = w,$$
$$(vuw) = v,\ (wvv) = v,\ \text{ and } (uwv) = p \neq w,$$

thus violating (A14). □

Every sun contains an induced fan (Fig. 6(a)), while a fan includes $K_{1,1,2}$. The $K_{1,1,2}$-free weakly median graphs are exactly the quasi-median graphs, which have only complete graphs as prime constituents. For the larger class of fan-free weakly median graphs, all prime members are included in hyperoctahedra.

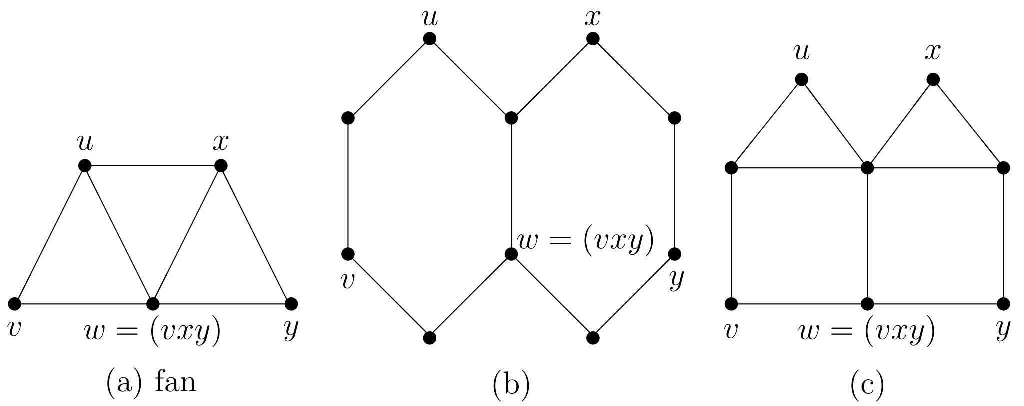


FIGURE 6. Graphs violating (A15)

Lemma 9. *An intrinsic algebra of a graph G satisfies the equation*

(A15) $\quad ((uv(vxy))(vxy)(xyv)) = (vxy)$

whenever G has unique quasi-medians and for any four vertices u, v, x, y either $(vxy) = (xyv)$ or $(uv(vxy)) = (vu(vxy))$ holds. In particular, cycle algebras, the house algebra, and the imprint algebra of the Petersen graph satisfy (A15).

Proof. The equation $((vu(vxy))(vxy)(xyv)) = (vxy)$ is equivalent to (A7) because setting $u = v$ returns (A7), and conversely, $(vu(vxy)) \in I(v, (vxy))$ and $(vxy) \in I(v, (xyv))$ hold in a graph with unique quasi-medians by [2, Proposition 4]. This proves the first assertion of the lemma. Note that any cycle algebra and the Petersen algebra fulfill the stronger property that (vxy) equals either v or (xyv). If in the house algebra, $(vxy) \notin \{v, (xyv)\}$, then $\{v, (vxy)\}$ constitutes a gated edge of the house, whence $(uv(vxy)) = (vu(vxy))$ for any vertex u. $\square$

The two graphs in Fig. 6(b),(c) are gated amalgams of two smaller graphs that both satisfy (A15). In either case, however, the composite graph violates (A15). This shows that gated amalgamation, even along congruence blocks, need not preserve equations.

Proposition 8. *For a graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is a fan-free weakly median graph;*
(ii) *every maximal induced subhyperoctahedron in G is a prefiber;*
(iii) *the imprint algebra of G satisfies* (A15).

Proof. (i) implies (ii): By [3, Lemma 3], a maximal induced subhyperoctahedron S is gated whenever it includes an induced 4-wheel. If S has no induced 4-wheel, then $S = I(u, v)$, where the common neighbors of u and v are pairwise adjacent. If S is not gated, then by [3, Lemma 1], there is a vertex x outside S adjacent to two vertices

$y, z \in S$. If y, z are different from u, v, we obtain one of the graphs of [2, Fig. 1(c,d)]. Otherwise, a fan occurs.

(ii) implies (i): The graphs of [2, Fig. 1(b,c,d)] and the fan contain an induced subgraph $K_{1,1,2}$ such that the fifth vertex u has at least two neighbors in this $K_{1,1,2}$, which therefore cannot be extended to a gated subhyperoctahedron since those four graphs are never included (as induced subgraphs) in subhyperoctahedra. If $K_{2,3}$ occurs as an induced subgraph of G, then any gated subhyperoctahedron S that contains two adjacent vertices w and x of this $K_{2,3}$ cannot include any further vertex of this subgraph. Then, however, either the pre-image $\psi_S^{-1}(w)$ or $\psi_S(x)$ is not convex, contradicting that S is a prefiber.

To establish the triangle condition (T), let u, v, w be a triplet as described in (T). Pick a gated subhyperoctahedron S that contains the two adjacent vertices v and w. Necessarily, the gate x of u in S is a common neighbor of v and w in $I(u, v)$, as required in (T). Finally, as for the quadrangle condition (Q), let u, v, w, z be a quartet as described in (Q). Assuming that $I(u, v) \cap I(u, w) = \{u\}$, we need to show that $d(u, z) = 2$. Take a subhyperoctahedron S that is a prefiber of G and contains the two adjacent vertices w and z. If v belongs to S, then $d(u, z) = 2$ is fulfilled. Therefore assume that v is outside S, whence z is the gate of v in S. The gate t of u in S either equals w or is a neighbor of w in $I(u, w)$. Then the vertex $x = (vtu) \in \ll t, u \gg \subseteq \psi_S^{-1}(t)$ is different from v. Since $d(v, t) = 2 + d(w, t)$, we infer that $d(v, x) = d(z, t)$ and $d(t, x) = 1$, yielding $x \in I(u, v) \cap I(u, w) = \{u\}$. If $t = w$, then $d(u, z) = 2$. Otherwise, $t \in N(w)$ and $d(u, z) = 3$. Consequently, $S = \ll t, z \gg$ is a 2-connected subhyperoctahedron. Now, by interchanging the roles of v and w, we may assume that $\ll s, z \gg$ is a 2-connected subhyperoctahedron for some common neighbor s of v and $u = x$. If $\ll s, z \gg$ and $\ll t, z \gg$ had a vertex $y \neq z$ in common, then it would follow $\ll s, z \gg = \ll y, z \gg = \ll t, z \gg$, a contradiction. Therefore $\ll s, z \gg \cap \ll t, z \gg = \{z\}$ and thus $z \in I(s, t)$, which however is in conflict with $d(s, t) \leq 2$. We conclude that (Q) is satisfied.

(i) implies (iii): If G includes an induced fan, with its vertices labelled as in Fig. 6(a), then

$$((uv(vxy))(vxy)(xyv)) = (uwx) = u \neq w = (vxy),$$

so that (A15) is violated.

(iii) implies (i): Conversely, four vertices in a prime constituent of a fan-free weakly median graph G either induce $K_{1,1,2}$ or are included in a K_4 subgraph or a decomposable subgraph (C_4 or $K_{1,2}$). Clearly, K_4 and $K_{1,1,2}$ meet the condition in Lemma 9 that is sufficient for (A15). $\square$

A stronger version of equation (A8), viz. (A16) below (alias axiom 4a of Isbell [12]), then characterizes the quasi-median graphs. These graphs can be defined as weakly modular graphs without induced $K_{2,3}$ and $K_{1,1,2}$ [8, 13], or alternatively, as weakly median graphs having bipartite intervals.

Proposition 9. *For a graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is a quasi-median graph;*

(ii) *every maximal complete subgraph of G is a prefiber;*

(iii) *some apex algebra of G satisfies one of the following (equivalent) equations*

(A16) $\quad (u(uwx)(vwx)) = (uwx)$,

(A16') $\quad ((vwx)u(uwx)) = (uwx)$.

Proof. (i) implies (ii): By [8, Theorem 1, (iii)⇒(iv)], every maximal complete subgraph of G is gated. Its gate map is a homomorphism by [2, Lemma 7] since quasi-median graphs are fiber-complemented [9].

(i) implies (iii): This follows from [8, Theorem 3].

(ii) implies (i): Clearly $K_{1,1,2}$ is a forbidden induced subgraph, whence every maximal subhyperoctahedron must be a complete graph. From Proposition 8 we then know that G is weakly median.

(iii) implies (i): The equations (A16) and (A16′) are equivalent because either one expresses that $(uwx) \in I(u, (vwx))$ for all $u, v, w, x \in V$. So, assume that (A16) holds for an apex algebra. Then (A8) holds as well. To show that G is apiculate, let v be a vertex from $I(u, w) \cap I(u, x)$ such that $I(u, v)$ is maximal with respect to inclusion. Then $(vwx) = v$ and hence $(uwx) \in I(u, v)$ by (A16), that is, $v = (uwx)$ is the u-apex relative to w and x. Suppose by way of contradiction that some interval $I(u, w)$ is not bipartite, that is, it contains two adjacent vertices v and x equidistant to u. It follows that

$$(u(uwx)(vwx)) = (uxv) \neq x = (uwx),$$

thus violating (A16). Hence G is a quasi-median graph. □

6. Axiomatics of discrete weakly median algebras

So far, we have derived the ternary algebras from specific graphs. Now, with the pool of equations at hand, we are able to reverse the association: starting from a "discrete" ternary algebra fulfilling certain equations as axioms one can recover the ternary algebra as the imprint algebra of an apiculate graph in which the intervals $I(u, v)$ are exactly the sets of elements x satisfying $(uvx) = x$. We say that a ternary algebra satisfying the axioms (A1), (A2), and (A3) is *discrete* if it does not contain an infinite bounded chain as a subalgebra; by a *bounded chain* we mean the median algebra associated with a linear order having a least element as well as a largest element. In particular, a finite chain is the imprint algebra of a path. Trivially, every intrinsic algebra of a (not necessarily finite) graph is discrete because any chain with bounds u and v is included in the interval $I(u,v)$ and hence has at most $d(u,v)+1$ elements.

In an abstract setting, an *interval space* $(V,\circ)$ [15] is a set V together with a binary set-valued operator $\circ$ that assigns to each pair of points a nonempty subset of V (called segment or interval) such that

$u,v \in u\circ v = v\circ u$ and $u\circ u = \{u\}$.

$(V,\circ)$ is *geometric* [15] if in addition

$w\in u\circ x$ and $v\in u\circ w$ imply $v\in u\circ x$ and $w\in v\circ x$.

To any ternary algebra on a set V satisfying the axioms (A1), (A2), (A3) (and hence (A1$'$) and (A3$'$) as well) one associates an interval space $(V,\circ)$: by virtue of (A3$'$) one can define

$u\circ v = \{(uvx) : x\in V\} = \{x\in V : (uvx) = x\}$,

so that $u,v\in u\circ v = v\circ u$ and $u\circ u=\{u\}$ follows from (A1),(A1$'$),(A2), and (A3).

Lemma 10. *Let $(V,\circ)$ be the interval space of a ternary algebra $\mathcal{A}=(V,(\ldots))$ satisfying* (A1), (A2), *and* (A3).

(a) *$w\in u\circ x$ and $v\in u\circ w$ imply $v\in u\circ x$ if and only if $\mathcal{A}$ satisfies*

(A17) $(uv(uwx)) = (ux(uv(uwx)))$.

In particular, (A17) *holds whenever* (A5) *does.*

(b) *$w\in u\circ x$ and $v\in u\circ w$ imply $w\in v\circ x$ if and only if $\mathcal{A}$ satisfies*

(A18) $(uwx) = ((uv(uwx))(uwx)x)$.

In particular, (A18) *holds whenever either* (A11) *or* (A16) *holds.*

(c) (A18) *implies* (A4$'$).

Proof. (a) If $w\in u\circ x$ and $v\in u\circ w$ for $u,x\in V$, then

$$v = (uvw) = (uv(uwx)) = (ux(uv(uwx))) = (uxv)$$

by (A17) and (A2), whence $v\in u\circ x$. Conversely, $(uwx)\in u\circ x$ and $(uv(uwx))\in u\circ(uwx)$ imply $(uv(uwx))\in u\circ x$, that is, (A17) holds, by the first part of the geometricity condition and (A2).

If (A5) is satisfied, then $(uv(uwx)) = (ux(uvw))$ by (A2), and therefore (A17) follows from (A3$'$).

(b) If $w \in u \circ x$ and $v \in u \circ w$ for $u, x \in V$, then

$$w = (uwx) = ((uv(uwx))(uwx)x) = ((uvw)wx) = (vwx)$$

by (A18) and (A2). Conversely, $(uwx) \in u \circ x$ and $(uv(uwx)) \in u \circ (uwx)$ imply $(uwx) \in (uv(uwx)) \circ x$, that is, (A18) holds, by the second part of the geometricity condition and (A2).

If (A11) holds, then we infer from $w \in u \circ x$ and $v \in u \circ w$ that

$$(vwx) = ((wuv)(uxw)x) = (w(uxw)(xu(wuv))) = (ww(xu(wuv))) = w.$$

Alternatively, if (A16) holds, we derive

$$(xwv) = (x(xuw)(vuw)) = (xuw) = w.$$

In either case, $w \in v \circ x$ is true.

(c) From (A18) we infer that $w \in u \circ x$ and $x \in u \circ w$ imply $w \in x \circ x = x$. Further, (A18) yields $(uvw) \in ((uvw)uv) \circ u$ because $(uvw) \in v \circ u$ and $((uvw)uv) \in v \circ (uvw)$. On the other hand, $((uvw)uv) \in (uvw) \circ u$ holds trivially. Hence $(uvw) = ((uvw)uv)$, that is, (A4′) holds. □

Discreteness carries over from a ternary algebra $\mathcal{A}$ to its corresponding interval space. To a discrete interval space $(V, \circ)$ one associates a graph $G = (V, E)$ by letting $uv \in E$ if and only if $u \circ v = \{u, v\}$. Recall that $(V, \circ)$ is called a *graphic interval space* [1, 15] if $u \circ v = I(u, v)$ holds for any $u, v \in V$, where I is the interval function of the graph G. The edges uv of G are retrieved from $\mathcal{A}$ by the condition $(uvw) \in \{u, v\}$ for all $w \in V$. In [1] we established that a geometric interval space is graphic whenever it satisfies the following *triangle condition:* for any three points u, v, w in V with

$$u \circ v \cap u \circ w = \{u\}, u \circ v \cap v \circ w = \{v\}, u \circ w \cap v \circ w = \{w\},$$

the intervals $u \circ v, u \circ w, v \circ w$ are edges of the underlying graph whenever at least one of them is an edge.

A subalgebra of an intrinsic algebra of a graph G is typically disconnected (taken as a subgraph) in G but may very well yield a graphic interval space in its own right. For instance, every metric triangle in G constitutes a subalgebra isomorphic to the imprint algebra of K_3.

Theorem 2. *The following statements are equivalent for a discrete ternary algebra* $\mathcal{A} = (V, (\ldots))$:

(i) $\mathcal{A}$ *is the imprint algebra of a weakly median graph* $G = (V, E)$;
(ii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), (A8), *and* (A11)*;*
(iii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), (A8′), *and* (A11)*;*
(iv) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A3), (A5), (A12′), *and* (A18).

In particular, any subalgebra of the imprint algebra of a weakly median graph is itself the imprint algebra of some weakly median graph. None of the axioms in (ii)-(iv) *are redundant.*

Proof. From Theorem 1 and Lemma 10 we know that the imprint algebra of a graph satisfies the equations listed in (ii) or (iii), respectively, if and only if the graph is weakly median. Therefore it remains to establish that a ternary algebra $\mathcal{A}$ satisfying one of the three sets of equations is the imprint algebra of a graph. First notice that (A1), (A2), (A5), and (A11) imply (A3). For $u, v, w, x \in V$,

$$\begin{aligned}
(wu(uwv)) &= (w(uwv)(vu(wuu))) && \text{by (A1),(A2)}\\
&= ((wuu)(uvw)v) && \text{by (A11),(A2)}\\
&= (uv(uvw)) && \text{by (A1),(A2)}\\
&= (u(uvv)w) && \text{by (A5)}\\
&= (uvw) = (uwv) && \text{by (A1),(A2).}
\end{aligned}$$

Hence (A3) holds. From Lemma 10 we conclude that in each case the interval space $(V, \circ)$ of $\mathcal{A}$ is geometric. Suppose by way of contradiction that the interval space $(V, \circ)$ violates the triangle condition for interval spaces. Then there exist adjacent vertices u, v and a vertex w in the graph G associated with $(V, \circ)$ such that $u \circ w \cap v \circ w = \{w\}$, but $u \circ w \neq \{u, w\}$. Then $(wuv) = w, (uvw) = u$, and $(vuw) = v$. Let x be a neighbor of u in $u \circ w$, i.e., a vertex such that $(uwx) = x$ and $u \circ x = \{u, x\}$. First assume that $\mathcal{A}$ satisfies the equations from (ii) or (iii). If $(xuv) = u$, then we get a contradiction to axiom (A8) and (A8′), respectively, because

$$((xuv)(wuv)x) = (uwx) = x \neq u = (xuv).$$

Therefore $u \notin x \circ v$. Since u is adjacent in G to both x and v, we conclude that $(xuv) = x$ and $(vux) = v$. Employing this in

$$((wux)(uvw)v) = (xuv) = x,$$

$$(w(uvw)(vu(wux))) = (wu(vux)) = (wuv) = w,$$

we see that (A11) is violated, giving a contradiction. Next assume that $\mathcal{A}$ satisfies the equations from (iii). Then u', v', w', and x' derived from u, v, w, and x as in (A12′) coincide with u, v, w, and x, respectively. Hence $(v'u'x')$ equals v or u. Consequently, the left-hand side of (A12′) is either u or w and hence cannot equal x, which is in conflict with (A12′). We conclude that the triangle condition is satisfied when (ii) or (iii) holds. Therefore $(V, \circ)$ is a graphic interval space. Moreover, the given ternary algebra is the imprint algebra of the underlying graph G of $(V, \circ)$. Indeed, let x be a vertex of $I(u,v) \cap I(u,w)$ such that $(uvw) \in I(u,x)$ and $I(u,x)$ is maximal with respect to the inclusion. Then we obtain $(uv(uwx)) = (uvx) = x$ and $(u(uvw)x) = (uvw)$, so that $x = (uvw)$ follows from (A5).

Finally, we will demonstrate that each axiom system in the theorem is irredundant.

Ad (A1): The constant ternary operation on $\{0,1\}$, defined by $(uvw) = 0$, satisfies all the equations listed in Theorem 2 except for (A1).

Ad (A2): The second and third ternary projections in $\{0,1\}$, defined by $(uvw) = v$ and $(uvw) = w$, respectively, satisfy (A1) and (A5) but violate (A2); moreover, (A8), (A8′), and (A11) are fulfilled by the second projection, whereas (A3), (A12′), and (A18) are fulfilled by the third projection.

Ad (A3): Take the integers $\mathbb{Z}_3 = \{0,1,2\}$ modulo 3, and define (uvw) as $u+1$ if $\{u,v,w\} = \{0,1,2\}$ and otherwise (when $|\{u,v,w\}| \leq 2$) via the majority rule. Then (A1), (A1′), and (A2) trivially hold. Consequently, (A3′) is fulfilled whenever $|\{u,v,w\}| \leq 2$. Since $(uvu{+}1)$ equals $u{+}1 = (uvw)$ for $\{u,v,w\} = \{0,1,2\}$, we infer that (A3′) is always true. In contrast, (A3) is violated because $(102) = 2$ but $(01(102)) = (012) = 1$. (A5) and (A18) easily follow from the valid equations (A1), (A1′), (A2), and (A3′) because at least two of the four variables u, v, w, x must be equal. In the case of (A12′), we infer that u', v', w', and x' all equal (uvw), so that this equation evidently holds.

Ad (A5): Consider the subalgebra $R_4 = \{u,v,w,x\}$ of the imprint algebra of $K_{1,1,3}$ as labelled in [2, Fig. 1(c)]. Then (A1)-(A4′) trivially hold but (A5) is violated [2, Proposition 1]. Further, (A8) and (A8′), (A11), and (A12′) are satisfied according to Examples 1(a), 2(a), and 3(b), respectively. (A18) holds because R_4 is a subalgebra of an imprint algebra.

Ad (A8) and (A8′): The imprint algebra of the house satisfies (A11) as well as (A1)-(A5) but violates (A8) and (A8′); see Example 2(b) and Lemma 6.

Ad (A11): The algebra $R_3 = \{t,u,v,w\}$ (see Fig. 4(c)) is a subalgebra of the imprint algebra of an apiculate graph and hence satisfies (A1)-(A5) but violates (A11) by Lemma 7. Since (A8) obviously holds whenever two of the four variables are equal or $(uwx) = u$, only one (up to symmetry) instance of (A8) needs explicit checking for R_3 :

$$((vwt)(uwt)v) = (tuv) = t = (vwt),$$

as required. Hence (A8′) holds as well.

Ad (A12′): The C_5 and house algebras trivially satisfy (A1)-(A5) and (A18) but violate (A12′) by Lemma 8.

Ad (A18): Modify the chain algebra of the linear order $0 < 1 < 2 < 3$ by turning $\{0, 1, 2\}$ into a K_3 algebra, that is:

$$(ijk) = \begin{cases} i & \text{if } \{i, j, k\} = \{0, 1, 2\}, \\ (i \wedge j) \vee (i \wedge k) \vee (j \wedge k) & \text{otherwise} \end{cases}$$

Then (A1)-(A3′) clearly hold. To check (A5) and (A12′), we may assume that $\{u, v, w, x\} = \{0, 1, 2, 3\}$. Then (A12′) readily follows because the only nontrivial case is when $\{u, v, w\} = \{0, 1, 2\}$ and $x = 3$. Indeed, in this case, $x' \in \{u, w\}$ and therefore $\{u', v', w', x'\} \in \{0, 1, 2\}$. As for (A5), we distinguish three cases. If $u = 3$, then all brackets $(\ldots)$ on both sides of (A5) are computed in the chain algebra as no bracket can contain the triplet $0, 1, 2$. If $w = 3$, then $(uwx) \in \{u, x\}$ and $(uvw) \in \{u, v\}$, so that either side of (A5) yields u because $(uvx) = u$. Finally, if 3 is one of v or x, say the latter, then $(uwx) \in \{u, w\}$ and $(uvw) = u$, so that again both sides of (A5) equal u. This concludes the proof of Theorem 2. □

Corollary 2. *The following statements are equivalent for a discrete ternary algebra* $\mathcal{A} = (V, (\ldots))$:

(i) $\mathcal{A}$ *is the imprint algebra of a quasi-median graph;*
(ii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), *and* (A16′);
(iii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A3), (A5), *and* (A16).

Proof. (i) implies (ii) by Proposition 9.

(ii) implies (iii): Using (A2), one derives (A1′) from (A16′) and (A2) by setting $u = v$ and $w = x$ in (A16′). Setting $v = w$ in (A16′) yields (A3) by virtue of (A1′). Then (A16) and (A16′) are equivalent because (A3) and (A2) hold.

(iii) implies (i): From Lemma 10 we infer that the algebra $\mathcal{A}$ satisfies (A8) (since (A16) and (A4′) hold) and that its interval space $(V, \circ)$ is geometric. We can therefore proceed as in the proof of the preceding theorem. In establishing the triangle condition, we can replace the argument involving (A11) by one using (A16) instead: for the triplet $u, v, w = (wvu)$ and the vertex $x = (vux) \in I(u, w)$ we get

$$w = (wvx) = (w(vux)(wux)) = (wux) = x,$$

a contradiction.

As to independence of axioms, note that (A16) and (A16′) are satisfied by the constant ternary operation and the third projection of $\{0, 1\}$ as well as by the imprint operation of R_4. Moreover, the 3-element algebra (defined above) that rejects (A3) satisfies (A16). This finishes the proof. □

Further axiomatic characterizations of the imprint algebras of quasi-median graphs can be found in [8, 12].

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